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CENTRE FOR DISTANCE EDUCATION

MANGALAGANGOTHRI - 574 199

DAKSHINA KANNADA DISTRICT, KARNATAKA STATE

COURSE 8

Pedagogy of School Subject - II (b)

MATHEMATICS

(Curriculum and Pedagogic Studies)

BLOCKS 1& 2

(PART - 1)

B.Ed. DEGREE PROGRAMME

(OPEN AND DISTANCE LEARNING)

SECOND YEAR B.Ed.

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(PART - 2)

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Contents

Overview of the Course

Mathematics is a skill-oriented subject. Unlike the social sciences, this subject needs to be addressed in parts giving equal importance to each of the topics it consists of. This paper tries to acquaint the student teacher with the idea of mathematics and the strategies of teaching this subject.

The first block nature, aims and objectives of Mathematics gives glimpses of the important working terms in mathematics and tries to introduce the subject's underlying. The objectives of mathematics are discussed giving separate importance to writing objectives in Arithmetic, Algebra, Geometry and Trigonometry. The second block focuses on Curriculum and instruction. Principles and Approaches to Curriculum construction are discussed, along with the approaches to teaching-learning mathematics. Strategies of teaching-learning mathematics are discussed in detail in this block. The third block is allotted for discussing the planning in teaching and learning mathematics. A fair amount of importance is given to ICT during the discussion of Learning Resources as it is the need of the hour. The fourth and the final black is kept aside for evaluation in mathematics. Evaluation forms an important part of teaching-learning and hence it is discussed in detail in this block. Diagnostic Testing is another important topic that is discussed in this block which will form an extra aid for a teacher in diagnosing the learning problems of students.

This paper covers all the essential knowledge for student teachers to teach Mathematics effectively. The student teachers need to decide the pedagogical approaches based on the nature of the topic, needs and background of students.

Block 1 : Nature, Aims and Objectives of Mathematics

Unit 1 : Nature and Scope of Mathematics

Unit Structure

1.1.1. Learning Objectives

After completing this Unit, the student teachers will be able to

- Explain the meaning of Mathematics;
- Explain the nature of Mathematics;
- Explain the Scope of Mathematics;
- List and elaborate different Branches of Mathematics; and
- Explain the uses and application of Mathematics.

1.1.2. Introduction

Mathematics finds a very important place in human progress and development. If we observe our day-to-day life, mathematics is inseparable from every walk of our life. Our simple daily activities like simple calculations, the proportion of items in our cooking, our monthly expenses, our body size, our normal breathing count, etc. need mathematics to keep track of it. Further, if we move on to some advance uses like our profession, mathematics is an integral part in a big way. Hence it can be understood that nothing in this world can work without the use of mathematics. Hence it is very essential that one understands and internalizes mathematics at a very early age. To understand mathematics, one should be aware of its nature and also its scope. This makes the learning of mathematics meaningful and useful. In this unit let us try to understand the meaning of Mathematics, its nature and scope.

1.1.3. Learning Points and Learning Activities

1.1.3.1. Meaning of Mathematics

Exercise 1

Write in the place below, everything that comes to your mind when you hear the word 'Mathematics'

I am sure concepts like, ‘counting, addition, subtractions, multiplication, and division’ and everything that you have learned about mathematics throughout your schooling must have crossed your mind. But have you any time thought of understanding the meaning of mathematics? Now let us see what is meant by mathematics by analyzing the definitions given by different mathematicians.

Etymologically the word ‘Mathematics has been derived from the Greek ‘mathema’ which means “Science, Knowledge or Learning”. Mathematics has been defined by different dictionaries as follows

- Mathematics is the science of number and space
- The science of measurement, quantity and magnitude
- Study of figures and numbers
- Study of patterns of structure, change and space.

The definitions indicate that Mathematics is an accepted science which deals with quantitative aspects of our life and knowledge.

Comte defined mathematics as “The science of indirect measurement”.
According to **Kant** “Mathematics is the indispensable instrument of all physical researchers”.

Gauss stated, “Mathematics is the Queen of Sciences and arithmetic is the queen of all mathematics.”

Bacon said, “Mathematics is the gateway and key to all sciences.”

Mathematics in the real sense is a science of space and quantity that helps us in solving the problems of life needing numeration and calculations. It provides an opportunity for the intellectual gymnastic of the man’s internal powers. It is an exact science and involves high cognitive abilities and powers.

Locke stated, “Mathematics is a way to settle in the mind a habit of reasoning.” A more comprehensive definition of mathematics was given by Courant and Robin when they defined mathematics in the following way. “Mathematics is an expression of the human mind which reflects the active will, the contemplative reason and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality.”

J.B. Shaw stated, “Mathematics is engaged, in fact, in the profound study of art and the expression of beauty.” According to Shaw, there are four significant methods of mathematics which give more insight into the nature of mathematics.

- Scientific, leading to generalizations of widening scope.
- Intuitive, leading to an insight into subtler depths.
- Deductive, leading to a permanent statement and rigorous form.
- Inventive, leading to the ideal element and creation of new realms.

Mathematics is therefore is not only ‘number work’ or ‘computation’, but is more about forming generalizations, seeing relationships and developing logical thinking and reasoning. The National Policy on Education (NPE-1986) stated “Mathematics should be visualized as the vehicle to train a child to think, reason, analyze and to articulate logically”. Mathematics should be shown as a way of thinking, art forms a beauty and as a human achievement.

Check Your Progress - 1

1. Mathematics is ________
    a) Science of number and space
    b) Study of numbers and figures
    c) Science of indirect measurements
    d) All of the above

2. According to Locke **“**Mathematics is a way to settle in the mind a habit of _________”
    a) Calculation
    b) Reasoning
    c) Simplification
    d) Analysis

1.1.3.2.Nature of Mathematics

Exercise II

In the space provided below, list the different features of mathematics that you have observed during your study of mathematics in school.

As you tried answering this question, all the processes of mathematics that you have during your schooling must have crossed your mind. Now let us try to make meaning out of these processes to ascertain the nature of mathematics.

Nature of Mathematics

1. **Mathematics as a Science of Discovery:** Mathematics is the discovery of relationships and the expression of those relationships in symbolic form – in words, in numbers, in letters, by diagrams or by graphs (E.E.Biggs, 1963). According to A.N. Whitehead (1912) “Every child should experience the joy of discovery.” Initially, a child’s discoveries may be observational. But later, when its power of abstraction is adequately developed, it will be able to appreciate the certitude of the mathematical conclusions that are drawn. This will give it the joy of discovering mathematical truths and concepts. Mathematics gives an easy and early opportunity to make independent discoveries.

2. **Mathematics as an intellectual game:** Mathematics can be treated as an intellectual game with its own rules and without any relation to external criteria. From this viewpoint, mathematics is mainly a matter of puzzles, paradoxes, and problem-solving – a sort of healthy mental exercise.

3. **Mathematics as the art of drawing Conclusions:** One of the important functions of school is to familiarize children with a mode of thought which helps them in drawing the right conclusions and inferences. According to the J.W.A. Young a subject suitable for this purpose should have three characteristics
    - That its conclusions are certain. At first, at least the learner must know whether or not he/she has drawn the correct conclusion.
    - That permits the learner to begin with simple and very easy conclusions to pass in well-graded sequence to very difficult ones, as the earlier ones are mastered.
    - That type of conclusions exemplified in the introductory subject be found in the other subjects also and in human interactions in general.

These characteristics are present in mathematics to a larger extent than in any other available subject.

4. **Mathematics as a tool subject:** It could be more elegantly expressed as mathematics, handmaiden to the sciences. From the beginning, down to the nineteenth century, mathematics has been assigned the status of a servant. Then in the nineteenth century, mathematics attained independence. It achieved completeness and internal consistency that it had not known before. Mathematics continued to be useful to other disciplines, but now it is dependent upon none of them. With its newfound freedom, mathematics established its own goals to pursue. Its mentors of the past – engineering, physical science and commerce –now became no more than its peers.

   Mathematics has its integrity, its beauty, its structure and many other features that relate to mathematics as an end in itself. However, many conceive mathematics as a very useful means to other ends, a powerful and incisive tool of wide applicability. John J Bowmen (1966) in an article titled "Mathematics and the Teaching of Sciences" stated. Now all students are captivated by the internal consistency of mathematics and for everyone who makes it a career, there will be dozens to whom it is only an elegant tool.

   According to Howard F. Fehr (1996), "If mathematics had not been useful, it would long ago have disappeared from our school curriculum as required study."

5. **Mathematics as a system of logical processes:** Polya suggested that mathematics has two faces. One face is a 'systematic deductive science'. This has resulted in presenting mathematics as an axiomatic body of definitions, undefined terms, axioms and theorems. Mario Pieri stated "Mathematics is a hypothetic deductive system" This statement means that mathematics is a system of logical processes whereby conclusions are deduced from certain fundamental assumptions and definitions that have been hypothesized. This has been reinforced by Benjamin Pierce when he defined mathematics as "The science which draws necessary conclusions." The student draws the inferences from the premises, provided the premises are true. In mathematics, granted the premises, the conclusion follows inevitably.

   Polya described the second face of mathematics by saying "Mathematics in the making appears as an experimental, inductive science. There has been a growing emphasis on the experimental side of mathematics. The generalizations follow as outcomes of the observations of mathematical phenomena and relationships. It is based on the principle that if a relationship holds good for some particular cases, it holds good for any similar case and hence the relationship can be generalized. Such a process is called inductive reasoning.

6. **Mathematics as an intuitive method:** Intuition implies the act of grasping the meaning or significance or structure of a problem without explicit reliance on the analytic apparatus of one's craft. It is the intuitive mode that yields a hypothesis quickly. It precedes proof; it is what the techniques of analysis and proof is designed to test and check. It is a form of mathematical activity that depends on the confidence in the applicability of the process rather than the importance of the right answers all the time.

   Intuition when applied to mathematics involves the concretization of an idea not yet stated in the form of some sort of operations or examples. A child forms an internalized set of structures for representing the world around him. These structures are governed by definite rules of their own. In the course of development, these structures change and the rules governing them also change in certain systematic ways. To anticipate what will happen next and what to do about it is to spin our internal models just a bit faster than the world goes. It is important to allow the student to express his intuition and check and verify its validity.

The first step in the learning of any mathematical subject is the development of intuition. This must come before rules are stated or formal operations are introduced. The teacher has to foster intuition in our young children, by following the right strategies of teaching.

Check Your Progress - 2
What is the nature of Mathematics?

1.1.3.3. Scope of Mathematics

Exercise III

Are you aware of different Branches of Mathematics? Then list the Branches of Mathematics you know and write its uses in the space provided below.

Branches of Mathematics	Uses

You will surely have included the branches of mathematics that you have studied in school i.e. Arithmetic, Algebra, Geometry, Statistics etc. These are just a tip of the ice-berg as compared to the vast number of branches that are there in Mathematics. The branches of mathematics and their uses form the Scope of Mathematics.

Let us understand the scope of mathematics under the following side headings

I. Fields of Study or Branches of Mathematics
II. Uses and Applications of Mathematics

I. Fields of Study or Branches of Mathematics
Let us see some of the important branches of Mathematics

Field of Study or Branches of Mathematics	The Scope of the Study
Arithmetic	A branch of pure mathematics devoted primarily to the study of the integers. It is sometimes called the 'The Queen of Mathematics' because of its foundational place in the discipline.
Elementary Algebra	Study of natural numbers and integers and their arithmetical operations. Whereas arithmetic deals with specified numbers, algebra introduces quantities without fixed values, known as variables.

Linear Algebra	Study of structure and space involving vectors and their structural properties.
Abstract Algebra	Intensive study of the properties of rational, real and complex numbers including rings and field structures.
Boolean Algebra	Boolean algebra is the branch of algebra in which the values of the variables are truth values true and false, usually denoted as 1 and 0 respectively. It has applications in designing digital circuits.
Computer Algebra	Branch of algebra relating to the study and development of algorithms and software for manipulating mathematical expressions and other mathematical objects.
Number Theory	Number theory mainly deals with the study of Natural Numbers. Number theorists study prime numbers as well as the properties of objects made out of integers (e.g. algebraic integers).
Lie Theory	The foundation of Lie theory is the exponential map of relating Lie Algebras to Lie groups. Lie theory has been particularly useful in mathematical physics.
Graph Theory	Branch of mathematics-related with emphasizing formalizing all the proofs in mathematical structures used to model pair-wise relations between objects.
Proof Theory	The branch of mathematics-related to emphasizing formalizing all the proofs in mathematics.
Order Theory	The field of mathematics extending the idea “for any two distinct real numbers, one must be greater than the other” to sets in general.
Knot Theory	The study of mathematical knots.
Game Theory	Game theory is “the study of mathematical models of conflict and co-operation between intelligent rational decision-makers.” It is mainly used in economics, political science and psychology, as well as logic, computer science, biology and poker.
Information Theory	Study of the quantification, storage and communication of information.
Probability Theory	Branch of mathematics that deals with quantities having random distributions.
Chaos Theory	Branch of mathematics that deals with complex systems whose behavior is highly sensitive to slight changes in conditions so that small alterations can give rise to strikingly great consequences.
Set Theory	Branch of mathematics dealing with the formal properties of sets as units and the expression of other branches of mathematics in terms of sets.

Geometry	Branch of mathematics concerned with questions of shape, size relative position of figures and properties of space.
Analytical Geometry	Analytical geometry applies methods of algebra to geometric questions, typically by relating geometric curves to algebraic equations.
Algebraic Geometry	Algebraic Geometry is a branch of mathematics, classically studying zeros of multivariate polynomials.
Convex Geometry	Convex geometry is the branch of geometry studying convex sets mainly in Euclidean space.
Discrete or combinational Geometry	The study of geometrical objects and properties either discrete or combinatorial, either by their nature or by their representation. It includes the study of shapes such as the platonic solids and the notion of tessellation
Differential Geometry	A mathematical discipline that uses the techniques of differential calculus, integral calculus, linear algebra and multi-linear algebra to study problems in geometry.
Hyperbolic Geometry	A non-Euclidean geometry coming into existence with the use of the negation of Euclid's parallel postulate.
Riemann Geometry	A Non-Euclidean geometry coming into existence with the use of the negation of Euclid's parallel postulate and infinite line postulate.
Coordinate Geometry	Study of Geometry using a co-ordinate system.
Differential Equations	A branch of mathematics involving mathematical equations that relates some function with its derivatives.
Calculus	The branch of mathematics-related to the mathematical study of change in the same way as geometry relates to the study of shapes.
Calculus of Variations	A field of mathematical analysis that deals with maximizing or minimizing functional, which are mappings from a set of functions to the real numbers.
Differential Calculus	The branch or area of calculus concerning rates of change and slopes of curves.
Integral Calculus	The branch or area of calculus concerning accumulation of quantities and the areas under and between curves.
General Topology	A branch of mathematics dealing with the basic set of theoretic definitions and constructions used in topology. It is also called the point-set topology. The main concept of the study is continuity, compactness and connectedness.
Differential Topology	Differential topology is the field dealing with differentiable functions on differentiable manifolds. It is closely related to differential geometry.

Algebraic Topology	A type of topology studying the properties of algebraic objects associated with a topological space and how these algebraic objects capture properties of such spaces.
Trigonometry	A branch of mathematics that studies relationships involving lengths and angles of triangles.
Spherical Trigonometry	Branch of spherical geometry that deals with the relationships between trigonometric functions of the sides and angles of the spherical polygons, especially spherical triangles.
Mathematical Analysis	Branch of mathematics dealing with limits and related theories, such as differentiation, integration, measure, infinite series and analytical functions.
Real Analysis	Branch of mathematical analysis dealing with the real numbers and real-valued functions of a real variable.
Complex Analysis	Branch of mathematical analysis that investigates functions of complex numbers.
Numerical Analysis	Study of Algorithms that use numerical approximation for the problem of mathematical analysis.
Operational Research	A discipline that deals with the application of advanced analytical methods, which helps to make better decisions.
Mathematics Programming	The use of a computer program to choose the best alternative from a set of available options. Mathematical programming uses probability and mathematical models to predict future events.
Discrete Mathematics	Discrete mathematics is the study of mathematical structure that is fundamentally discrete rather than continuous. In contrast to real numbers that have the property of varying smoothly, the objects studied in discrete mathematics – such as integers, graphs, and statements in logic- do not vary smoothly in this way, but have distinct separated values.
Statistics	The practice or science of collecting and analyzing numerical data, large quantities, especially to infer proportions in a whole form those in a representative sample.
Biostatistics	A branch of mathematical statistics applied in the study of facts and principles of biology.
Statistical Quality Control	Statistical quality control refers to the use of statistical methods in the monitoring and maintaining of the quality of products and services. Branch of mathematics dealing with the application of mathematics to problems in physics and the development of mathematical methods suitable for such applications and the formulation of physical theories.
Mathematical Physics	Branch of mathematics dealing with the application of mathematics to problems in physics and the development of mathematical methods suitable for such applications and the formulation of

	physics theories
Mathematical Chemistry	Branch of mathematics dealing with the novel applications of mathematics to chemistry; it concerns itself principally with mathematical modeling of chemical phenomena.
Mathematical Biology	Mathematical biology is an interdisciplinary scientific research field with a range of applications in biology, biotechnology and medicine.
Mathematical Statistics	Mathematical statistics is the application of mathematics to statistics involving the use of techniques such as mathematical analysis, linear algebra, stochastic analysis, differential equations and probability theory.
Cryptography	Cryptography is the practice and study of techniques for providing information security by preventing third parties or the public from reading private messages. Applications of cryptography include ATM cards, computer passwords and electronic commerce.
Combinatorics	Branch of mathematics concerning the study of finite or countable discrete structures.
Logics	Branch of mathematics consisting of a system or set of principles underlying the arrangements of elements in a computer or electronic device to perform a specified task.
Computational Mathematics	Computational Mathematics involves mathematical research in areas of science where computing plays a central and essential role, emphasizing algorithms, numerical methods and symbolic computations.
Mathematical model	Branch of mathematics dealing with the method of simulating real-life situations with mathematical equations to forecast their future behavior.

II. Uses and Applications of Mathematics

Let us see the scope of mathematics in terms of its uses and applications of Mathematics.

1. **Mathematics and progress of society:** An important yardstick to measure the scope of mathematics may be in the form of drawing a picture of its contribution to the overall progress and contributions in society. Right from the earliest of civilizations to the present age of computerized progress, we may witness that the study of mathematics has always been the backbone of such progress. Trade and commerce, banking and financing, global marketing, construction and architectural progress, transportation and communication, a scientific and technological development that we have at our disposal are all the outcomes of the progress and development in mathematics which we are having today with us.

2. **Mathematics and study of other subjects:** The evidence about the largeness of the scope of mathematics in the realm of school education may be witnessed by looking at the uses and applications of mathematics in the teaching-learning of other subjects of the school curriculum. The studies of the subjects like languages, sciences including social sciences, work experience, health and physical education, art and painting, etc are all

quite helped and facilitated through the study of mathematics. We will be dealing with this issue in detail in chapter four of this text.

3. **Mathematics and avenues of career development:** The scope of the study of the subject mathematics has been so widened and enlarged that now the key of all the important professions and avenues of career development lies in the art and skills related to mathematics. In general, entry, as well as success in the professions and careers like below, may be properly availed through the study of mathematics.

Careers in Mathematics

Career-related to

- Engineering
- Banking and Commerce
- Finance and Accountancy
- Economics and Business
- Research
- Teaching and Instructional Work
- Statistics
- Astrology, Astronomy and forecasting

Profession in Mathematics

- Teaching job in schools and higher academic institutions –colleges and universities.
- Services or consultancy in Banking institutions and commercial establishments.
- Jobs related to Engineering (after going through one or the other Engineering professional courses)
- Economist, auditor, tax consultant and corporate financial advisor.
- System analyst and operation research analyst.
- Statistician
- Research Scientist and Data Analyst
- Environmental Mathematician and Ecologist
- Geophysical Mathematician
- Chemical/physical/biological mathematician
- Computer scientist, space scientist and robotics engineer
- Astronomer and Astrologer
- Financial estimate makers and forecasting personnel
- Investment Advisors and planners.

Check Your Progress - 3

1. List the different Branches of Mathematics.
2. What are the uses of Mathematics?

1.1.4. Let us Summarise

In the first part, we attempted to understand the meaning of Mathematics. Definitions of mathematics given by different philosophers have been discussed. We found that "Mathematics is a way of thinking, an artand a human achievement. In the second part of the unit, we discussed the **Nature of mathematics** in the following contexts.

- Mathematics is a Science of Discovery
- Mathematics is an intellectual game
- Mathematics is the art of drawing Conclusions

- Mathematics is a tool subject
- Mathematics is a system of logical processes
- Mathematics is an intuitive method

A list of main (famous) branches of Mathematics together with their scopes have been listed, to recall a few, we name Arithmetic, Elementary Algebra, Linear Algebra, Abstract Algebra, Boolean Algebra, Computer Algebra, Number Theory, Field Theory, and many more. In the last part of the unit, main Areas of Application of Mathematics have been discussed and they are:

- Mathematics in the progress of society
- Mathematics in the study of other subjects
- Mathematics and avenues of career development

1.1.5 Answers to 'Check Your Progress – 1, 2 and 3'

Check Your Progress - 1

1. d) All of the above
2. b) Reasoning

Check Your Progress - 2
Nature of Mathematics

1. Mathematics is a Science of Discovery
2. Mathematics is an intellectual game
3. Mathematics is the art of drawing Conclusions
4. Mathematics is a tool subject
5. Mathematics is a system of logical processes
6. Mathematics is an intuitive method

Check Your Progress - 3
Branches of Mathematics

Arithmetic, Elementary Algebra, Linear Algebra, Abstract Algebra, Boolean Algebra, Computer Algebra, Number Theory, Lie Theory, Graph Theory, Proof Theory, Order Theory, Knot Theory, Game Theory, Information Theory, Probability Theory, Chaos Theory, Set Theory, Geometry, Analytical Geometry, Algebraic Geometry, Convex Geometry, Discrete or combinational Geometry, Differential Geometry, Hyperbolic Geometry, Riemann Geometry, Co-ordinate Geometry, Differential Equations, Calculus, Calculus Variations, Differential Calculus, Integral Calculus, General Topology, Differential, Topology, Algebraic Topology, Trigonometry, Spherical Trigonometry, Mathematical Analysis, Real Analysis, Complex Analysis, Numerical Analysis, Operational Research, Mathematics Programming, Discrete Mathematics, Statistics, Biostatistics, Statistical Quality Control.

Uses of Mathematics

- Mathematics in the progress of society
- Mathematics in the study of other subjects
- Mathematics and avenues of career development

1.1.6. Unit end Exercises

1. Elucidate the meaning of mathematics stating definitions of various mathematicians.
2. Explain the nature of Mathematics.
3. Explain the Scope of Mathematics
4. What are the different branches of Mathematics? Explain its scope.
5. Elaborate on the application of Mathematics.

1.1.7. References

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Block 1 : Nature, Aims and Objectives of Mathematics

Unit 2 : Meaning and Building Blocks of Mathematics - Undefined Terms, Definitions, Axioms, Theorems

Unit Structure

1.2.1. Learning Objectives

After completing this Unit, the student teachers will be able to

- Explain the Building Blocks of Mathematical System;
- Explain the Undefined Terms of Assumptions;
- Explain the terms - Definition, Postulates, Propositions, Axioms, Hypothesis, Open Questions, Quantifiers;
- Explain the meaning of Mathematical Theorem; and
- Explain the variants of the mathematical theorem.

1.2.2. Introduction

Mathematics has an elaborate process that makes it useful and applicable to every walk of our lives. However, to understand this process several simpler processes work in unison to make mathematics what it is. Mathematics begins with assumptions that have no proofs as such but can be believed because of its existence and proceeds towards proofs which can be established with logical deductions. Each process of mathematics holds the importance of its own and needs to be understood for greater assimilation. In this unit, we shall try to understand the various terms used in mathematics which are the building blocks of this subject.

1.2.3. Learning Points and Learning Activities

1.2.3.1. Building Blocks of Mathematical System-1

Exercise

Recall your mathematics classes as a student and list all the technical terms that you remember which you came across as you studied Mathematics. Also, write what you understood about them.

As you tried answering the question, I am sure you remembered at-least a few such as the Theorem, Axioms and Postulates. Have you then tried to understand what they stood for? Now let us understand some important terms which are the building blocks of a mathematical system.

Undefined terms of assumptions, definitions, postulates, axioms, propositions (including their composite forms like the conjunction, disjunction, implication, equivalence and negation), proofs, open sentences, quantifiers, mathematical theorem and its variants like converse, inverse and contra positive, methods of giving proof for the mathematical theorems etc are termed as the building blocks of a mathematical system.

A. Undefined Terms of Assumptions

In a system of mathematics, there may exist many terms that may be accepted and used with no attempts of getting them defined in one or the other way. It happens on a simple logic that it is not possible to define everything and one has to start from something that goes undefined with a clear-cut assumption of its existence. They are assumed in the way they appear, exist or made into use. For example, in geometry the terms, point, line and surface and algebra, set, number and variable have been accepted as undefined or primitive terms. Similarly, in arithmetic the undefined and primitive terms are "0", "number" (in the sense of non-negative whole number) and "successors" (in the sense that $x + 1$ is the successor of$x$). It should also be made clear that mathematicians are free to fix undefined terms according to the system.

Definitions:

Starting from the undefined terms we may now define some technical terms. The help of non-technical language may also be taken for this task. These defined terms will then help in defining more technical terms and so on.

Example: Right angle triangle in that triangle in which one of its angle measures as 90^0. (This definition is based on the already defined terms like angle, triangle, right angle etc).

B. Postulates:

Generally, in proving a theorem the help of some already proved theorem or theorems are taken. Going backward there must be a theorem for which we have no previous theorem to help. In proving this first theorem we have to accept certain statements without stressing their proofs. These statements, which are accepted as true in a particular system without stressing their proofs, are labeled as postulates. Meaning thereby that every system has its selection of postulates. i.e., the unproved first principles. For example, in the case of a system in arithmetic adopting "0" (zero), "number" and "successor" as the undefined or primitive terms, the related postulates may be named as below:

1. 0 is a number
2. Every member has a number as its unique successor.
3. Two members having the same successors are identical.
4. 0 is not a successor of any number
5. If 0 belongs to a class F, and if whenever a number x belongs to F, the successors of x belong also to F, then all numbers belong to F.

Similarly, in the case of elementary geometry, the related postulates may be named as below.

1. A line is a set of points containing at least two points.
2. A plane contains at least three distinct non-collinear points.
3. If two planes contain a point in common, they have another point in common.

In any system of geometry Euclidean or Non-Euclidean postulates make it possible to prove several theorems with great rigor and with no reference to drawings. Every system in all the branches of mathematics is known to have its own sets of undefined terms (assumption), and postulates. However, in the selection of postulates for a system two things should be kept in mind.

1. Postulates should be totally independent
2. There should not be any internal inconsistency.

C. Propositions:

According to Webster's Seventh New Collegiate Dictionary (1970:684), a proposition may be defined as something proposed or offered for consideration or acceptance. In its practical shape or appearance, a proposition stands for a grammatically correct meaningful sentence in the form of a declarative statement or proposal put forward for being accepted or rejected by terming it either true or false, but not both. If a proposition is true we say that its truth value is true and if it is false we say its truth value is false.

Examples

1 The sum of the three angles of a triangle is two right angles is a proposition (declarative statement) which is true.

2 In an equilateral triangle, one angle is right angle is also a proposition (another declarative statement) which is false. However the statement: Are three sides of an equilateral triangle equal? Is not a proposition, because it is not a declarative sentence as it does not declare anything but is only an interrogation.

A question here may arise that how can we say that a particular proposition is true or false? For doing so, we make use of a process known as logical reasoning involving an inductive-deductive approach. In technical terms, it is known as providing proof for the establishment of the truth of proposition. Generally for proving proposition, we try to take help from the already established and proved proposition for collecting essential material/data and then apply the rules of reasoning for arriving at the truth of the given proposition. However, there may also arise several situations where we have to make use of one or the other axioms along with the previously established propositions for providing proof of a particular proposition.

D. Axioms

According to Webster's Seventh New Collegiate Dictionary (1970:62), an axiom is nothing but a proposition regarded as a self-evident truth. In using logical reasoning for proving a proposition, we use axioms as self-evident truth with no need of getting them proved.

Historically the term 'axiom' was first used by the Great philosopher Aristotle. According to him, every demonstrative science must start from an indemonstrable principle known as the first principle. Among these first principles some are peculiar to the particular science but others are common to all sciences. It is these latter first principles common to all sciences which may be termed as axioms. In other words, axioms may be known as the common opinions or notions from which all demonstration proceeds and as these things which anyone must be told who is to learn anything at all.

In the 'Elements' the book written by Euclid these first principles are listed in two categories, the postulates and the common notions. The former are the principles peculiar to the particular science of geometry and the latter, the common notions are evidently the same as Aristotle's axioms. Proclus who has written comments on the 'First Book' of Euclid tells us exclusively that the two terms Euclid's 'common notions' and Aristotle's 'axioms' are synonymous.

Historically, therefore, there stands a clear cut distinction between the terms axioms and postulates. However, a little confusion has been created by some mathematicians in modern times by using the term 'postulates' and 'axioms' as synonymous. But in any way, it is quite advantageous to maintain a distinction between these two terms by keeping reserved the term "axioms" for the axioms of logic or common notions and to use "postulates" for those assumptions or first principles (beyond the principles of logic) by which a particular mathematical discipline (say geometry) may be defined.

E. Hypothesis

In the process of seeking the answer to a raised question or finding the solution to a problem, through deductive reasoning one may take the help of some imaginable answers or solutions (in the form of one or the offer proposition) for being tried or tested true or false. Such imaginable, assumed and guessed answers, solutions or propositions may be termed as hypotheses.

Webster's Seventh New Collegiate Dictionary (1970:410) also supports such meaning of the term hypothesis by defining it as "a tentative assumption made to draw out and test its logical or empirical consequences."

F. Open Questions

In the case of defining a proposition, we have said that it stands for a declarative statement, expressed in a meaningful sentence that can either be proved true or false and not both true and false in anyway. However, there may be many situations arising in the study of mathematics and also in our day to day life where we may be confronted with or have to make use of the declarative statements expressed in meaningful sentences that can be proved both true or false (true in some situations and false in the other). These sentences are referred to as open sentences and stand as quite free and open for being accepted, rejected or accepted as well as rejected in the or the other circumstances.

As an example of such an open sentence let us name the sentences (i) $x^2 + 3x + 2 = 0$ and (ii) $x > 1$. These carry the unique distinction of being true as well as false for certain values of x. For example, in case x is given the value 1, then we may see that these can be proved as false, but if we assign the value -1 to x, then the sentences are termed as true.

Thus, the sentences $x^2 + 3x + 2 = 0$ or $x > 1$ become propositions on assigning real values to the unknown x. Inset-language (explained in this chapter ahead) the set of all values which the unknown in the sentence is allowed to take is known as the universal set. In the language of mathematics, thus an open sentence is a sentence involving a variable (such as $x, y,$ etc) and which becomes a proposition (proved to be true or false and not both true and false) on situation of values for the variable from the universal set. In the language of mathematics, such open sentences are denoted by symbols p_x, q_x, etc. As other examples of these open sentences, we may name the algebraic expressions, like, $x^2 + 3x = (2x + 1)(x + 1)$, $y^3 - 3y^2 = 5$and$x^2 + 3x + 2 > 5$, etc.

Check Your Progress- 1

Explain the concepts postulates, propositions and axioms

1.2.3.2. Building Blocks of Mathematical System-2

A. Quantifiers

We have seen that declarative statement, sentence or expression like $x^2 + 3x + 2 = 0$ or $x > 1$ with variables (in case *x*) is not a proposition. It is an open sentence. It can be true or false depending on the value of x.

For turning either of these two, $x^2 + 3x + 2 = 0$ or $x > 1$ into a proposition we need to apply one of the following two operations to the variable or variables used in the expression (open sentence).

1. Assigning a value to the variable such as 1, -1 etc
2. Quantifying the variable using a quantifier.

Let us see what these quantifiers are?

As a matter of definition, a language element- word or symbol which generates quantification to the expression of a statement is called a quantifier.

There are two types of quantifiers: universal quantifier and existential quantifier.

While universal quantifiers provide universal quantification, existential quantification is provided through existential quantifiers.

B. Universal Quantifiers

When we make use of the words like 'for all', 'for every', 'for each' in the statements, then the use of these words provide universal quantification t the given expression such as below.

1. All the students in this school are required to wear the prescribed dress.
2. The sum of the interior angles f every quadrilateral is equal to 360^0.
3. For every real number x, $(x + 1)^3 = x^3 + 3x^2 + 3x + 1$
4. All even numbers greater than 2 are composite.

In these declarative statements universe under which the given statement holds valid is clearly defined such as 'school', 'quadrilateral', 'real numbers', 'even numbers' etc and the word. 'All', 'every' etc. used in the sentences are performing the task f quantifying the expressions in a proper way.

C. Existential Quantifiers

We also, make use of words like 'there is a', 'for some', etc in our expression of a declarative statement, the use of these words provide existential quantification to the given expression such as below.

1. For some integer $x, x^2 = 4$
2. There is a rational number whose square is 3.
3. These are some students in the class who have not studied English in the previous classes.
4. There exists a rhombus that is not a square.

Until now we have provided examples of all logical quantifiers (universal as well existential) by making the use of the words, written and spoken in the natural language.

However, mathematics as we know it is distinguished to have its language of symbols and abstraction. Hence we have to make use of the symbols below for the expression of logical quantifiers.

- The symbol for the universal quantifier ‘all’, ‘every’ is ‘∀’, a rotated letter ‘A’.
- The symbol for the existential quantifiers “there exists a”, “some” etc is “∃”, a rotated letter ‘E’.

D) A Mathematical Theorem and Its Variants

A system of mathematics is very much characterized by the presence of mathematical theorems. As a matter of definition, we can term a mathematical theorem as a “logical valid conclusion drawn from a set of premises, axioms and already established theorems of mathematics system” (NCERT, 2012:22)

In mathematics, the theorem is a non-self-evident statement that has been proven to be true, either based on generally accepted statements such as axioms or based on previously established statements such as other theorems. A theorem is hence a logical consequence of the axioms, with a proof of the theorem being a logical argument that establishes its truth through the inference rules of a deductive system. As a result the proof of a theorem is often interpreted as justification of the truth of the theorem statement. In light of the requirement that theorems be proved, the concept of a theorem, the concept of the theorem is fundamentally deductive, in contrast to the notion of a scientific law, which is experimental.

As an example of such mathematical theorems in the mathematical system involving geometrical grounds, we can name the following.

In a triangle, if two sides are equal, then the angles opposite to equal sides are also equal.

In the above example, the first half of the statement of the theorem may be termed as the premise on which the second half of the statement in the form of a conclusion is based. In the mathematical language of symbol if we name the first half of the statement as *‘p’* and the second half as *‘q’* then the statement of a theorem can be diagrammatically represented as

$p \rightarrow q$ *(p implies q)*

Variants of Mathematical Theorem

A mathematical theorem $p \rightarrow q$ can be found to have its variants in the form of

1. Converse of the theorem $(q \rightarrow p)$
2. Inverse of the theorem $(\neg p) \rightarrow (\neg q)$
3. Contra positive of the theorem $(\neg q) \rightarrow (\neg p)$

i. Converse of the Theorem $(q \rightarrow p)$

A converse of the given theorem is obtained by getting its premise (p) and the conclusion (q) changed. Let us see what will be the converse of the theorem (given above by us as an example) if we interchange its premise and conclusion. It will be changing in the manner below.

Converse: In a triangle, if the angles opposite to two sides are equal, then these two sides are also equal.

ii. Inverse of the theorem $(\neg p) \rightarrow (\neg q)$

Inverse of a statement is obtained by providing its negation. Therefore in the inverse of a theorem, we may say that the premise as well as conclusion. (The first and second halves of the theorem) are stated in terms of their negation. In this way, the inverse of the theorem in our example will appear in the manner given below.

Inverse: In a triangle, if two sides are not equal, then the angles opposite to these two sides are not equal.

iii. Contra positive of the Theorem $(\neg q) \rightarrow (\neg p)$

In having contrapositive of given theorem two tasks are simultaneously performed to the premise and conclusion parts of the statement of the theorem, first to introduce negation and then have an inversion of these negations. In this way, contra positive of a theorem is obtained by replacing its premise by the negation of its conclusion and the conclusion by the negation of its premise.

As a result, the statement of the theorem in the example will be changed in the manner given below

Contrapositive: In a triangle, if the angles opposite to two sides are not equal, then the corresponding two sides are not equal.

Check Your Progress- 2
List the different building blocks of Mathematics.

1.2.4. Let us Summarise

- **Building Blocks of Mathematical System**
  - **Undefined Terms Of Assumptions:** Undefined terms in mathematics which are accepted and used on the logic that it is not possible to define everything and one has to start from something that goes undefined with a clear cut assumption of its existence.
  - **Definitions:** Technical terms that are defined using the undefined terms.
  - **Postulates:** The statements which are accepted as true in a particular system without stressing on their proofs are labeled as postulates.
  - **Propositions:** A proposition may be defined as something proposed or offered for consideration or acceptance
  - **Axioms: A**n axiom is a proposition regarded as a self-evident truth.
  - **Hypothesis:** a tentative assumption made to draw out and test its logical or empirical consequences.
  - **Open Questions:** There are situations arising in the study of mathematics and also in our day to day life where we may be confronted with or have to make use of the declarative statements expressed in meaningful sentences that can be proved both true or false. These sentences are referred to as open sentences and stand as quite free and open for being accepted, rejected or accepted as well as rejected in the or the other circumstances.
  - **Quantifiers:** There are two types of quantifiers: universal quantifier and existential quantifier.
  - While universal quantifiers provide universal quantification, the existential quantification is provided through existential quantifiers.
  - **Mathematical Theorem:** Logical valid conclusion drawn from a set of premises, axioms and already established theorems of mathematics system.
  - **Variants of Mathematical Theorem:** A mathematical theorem $p \rightarrow q$ can be found to have its variants in the form of
    1. Converse of the theorem $(q \rightarrow p)$
    2. Inverse of the theorem $(\neg p) \rightarrow (\neg q)$
    3. Contra positive of the theorem $(\neg q) \rightarrow (\neg p)$

1.2.5. Answers to 'Check Your Progress – 1'

Check Your Progress- 1
Refer section 1.2.3.1

Check Your Progress - 2
Building Blocks of Mathematical System

4. Undefined Terms Of Assumptions
5. Definitions
6. Postulates
7. Propositions
8. Axioms
9. Hypothesis
10. Open Questions
11. Quantifiers
12. Mathematical Theorem
13. Variants of Mathematical Theorem

1.2.6. Unit end Exercises

1. Name the building blocks of Mathematical System and explain them briefly.
2. Which are the different undefined terms? Explain
3. What are definitions? Elaborate with examples
4. Elucidate Axioms.
5. What is a Hypothesis? Explain with examples.
6. Explain Theorems and their variants.
7. Explain an open question.
8. Explain different types of quantifiers
9. Explain the different variants of a theorem.

1.2.7. References

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Block 1 : Nature, Aims and Objectives of Mathematics

Unit 3 : The Nature of Mathematical Propositions

Unit Structure

1.3.1. Learning Objectives

After completing this Unit, the student teachers will be able to

- Explain the nature of mathematical truth;
- Deduce Addition and Multiplication form Peano's Axioms;
- Explain the basic building blocks of logic;
- Know the laws of logic, logical equivalence; and
- List the rules of inferences and use them to check the validity of the given arguments.

1.3.2. Introduction

We live in a world where our notions and understanding of the world is very much shaped and limited by what we perceive. Mathematics deals with abstract notions in an essentially systematic (logically precise) way. Mathematics has been quite successful in developing our understanding of the world. A study into the nature of mathematical objects-its objectivity-could be useful in providing insights into the nature of the physical world. An investigation into the concept of mathematical truth - absolute, self-evident aspect, and axiomatic deductive nature might provide understanding about implications of mathematical propositions.

It is a basic principle of scientific inquiry that no proposition and no theory is to be accepted without adequate grounds. In empirical science, which includes both the natural and the social sciences, the grounds for the acceptance of a theory consist in the agreement of predictions based on the theory with empirical evidence obtained either by experiment or by systematic observation. But what are the grounds which sanction the acceptance of mathematics? That is the question proposed to discuss in this unit.

1.3.3. Learning Points and Learning Activities

1.3.3.1. The Nature of Mathematical Propositions -1

1. The Nature of Mathematical Truth

One of the several answers which have been given to the problem of acceptance of mathematics asserts that the truths of mathematics. This is in contradistinction to the hypotheses of empirical science, require neither accurate evidence nor any other justification because they are "self-evident." This view, however, which ultimately demotes decisions as to mathematical truth to a feeling of self-evidence, encounters various difficulties.

First of all, many mathematical theorems are so hard to establish that even to the specialist in the particular field they appear as anything but self-evident. Secondly, it is well known that some of the most interesting results of run counter to deeply ingrained intuitions and the customary kind of feeling of self-evidence. Thirdly, the existence of mathematical conjectures such as those of Goldbach conjecture, Twin prime conjecture, and of Four colour theorem, which is quite elementary in content and yet undecided up to this day. This certainly shows that not all mathematical truths can be self-evident. And finally, even if self-evidence were attributed only to the basic postulates of mathematics, from which all other mathematical propositions can be deduced, it would be pertinent to remark that judgments as to what may be considered as self-evident are subjective; they may vary from person to person and certainly cannot constitute an adequate basis for decisions as to the objective validity of mathematical propositions.

2. The Analytic Character of Mathematical Propositions

According to another view, advocated especially by John Stuart Mill, mathematics is itself an empirical science which differs from the other branches such as physics, chemistry, botany etc., mainly for two reasons: (a)Its subject matter is more general than that of any other field of scientific research and(b) its propositions have been tested and confirmed to a greater extent than those of even the most firmly established sections of physics or chemistry. According to this view, the degree to which the laws of mathematics have been borne out by the past experiences of mankind is so impressive that, we have come to think of mathematical theorems as qualitatively different from the well-confirmed hypotheses or theories of other branches of science. We consider them as definite, while other theories are thought of as at best "very probable" or very highly confirmed.

Consider now a simple "hypothesis" from arithmetic: 4 + 3 = 7. If this is actually an experiential generalization of past experiences, then it must be possible to state what kind of evidence would force us to concede the hypothesis was not generally true after all. If any disconfirming evidence for the given proposition can be thought of, the following illustration might well be typical of it: We place some germs on a slide, putting down first four of them and then another three. Afterwards, we count all the germs to test whether in this instance 4 and 3 actually added up to 7. Suppose now that we counted 8 germs altogether. Do we consider this as experientially the false? Clearly not; rather, we would assume we had made a mistake in counting or that one of the germs had split in two between the first and the second count. It simply states that any set consisting of 4 + 3 objects may also be said to consist of 7 objects. And this is so because the symbols "4 + 3" and "7" denote the same number: they are synonymous because the symbols "4," "3," "7," and "+" are defined (or tacitly understood) in such a way that the above identity holds as a consequence of the meaning attached to the concepts involved in it.

The statement that 4 + 3 = 7, then, is true for similar reasons as, say, the assertion that no sexagenarian is 45 years of age. (Note that a sexagenarian is a person who is between 60 and 69 years old). Both are true simply by virtue of definitions. Statements of this kind share certain important characteristics: Their validation naturally requires no experimental evidence. In the language of logic, sentences of this kind are called analytic or independent of experience.

3. Mathematics is an Axiomatic Deductive System

As argued so far that the validity of mathematics rests neither on its unproven self-evidential character nor on any experiential basis, but derives from the conditions which determine the meaning of the mathematical concepts, and that the propositions of mathematics are therefore essentially "true by definition." This latter statement, however, is oversimplified and needs restatement and a more careful justification. For the rigorous development of a mathematical theory proceeds not simply from a set of definitions but rather from a set of non-definitional propositions which are not proved within the theory; these are the postulates or axioms of the theory. They are formulated in terms of certain basic or primitive concepts for which no definitions are provided within the theory. It is sometimes asserted that the postulates themselves represent "implicit definitions" of the primitive terms.

Once the primitive terms and the postulates have been laid down, the entire theory is completely determined; it is derivable from its basis of postulates in the following sense: Every term of the theory is definable in terms of the primitives, and every proposition of the theory is logically deducible from the postulates. To be entirely precise, it is necessary also to specify the principles of logic which are to be used in the proof of the propositions, i.e., in their deduction from the postulates. These principles can be stated quite explicitly. They fall into two groups: **Primitive sentences, or Postulates** of logic (such as: If p and q is true, then certainly p is true), and **Rules of Deduction or Inferences.**

4. The Peano's Axiom System as a Base

Let us now consider a postulate system from which the entire arithmetic of the natural numbers can be derived. This system was devised by the Italian mathematician and logician G. Peano (1858-1932). The primitives of this system are the terms "0" "number," and "successor." While, of course, no definition of these terms is given within the theory, the symbol "0" is intended to designate the number 0 in its usual meaning, while the term "number" is meant to refer to the whole numbers 0, 1, 2, 3, . . . exclusively. By the successor of a natural number *n*, which will sometimes briefly be called *n'*, is meant the natural number immediately following *n* in the natural order. Peano's system contains the following 5 postulates:

P1. 0 is a number
P2. The successor of any number is a number
P3. No two numbers have the same successor
P4. 0 is not the successor of any number
P5. If **P** is a property such that (a) 0 has the property **P**, and (b) whenever a number *n* has the property **P**, then the successor of *n* also has the property **P**, then every number has the property **P**.

The last postulate symbolizes the principle of mathematical induction and illustrates in a very obvious manner the enforcement of a mathematical "truth" by stipulation. The construction of elementary arithmetic on this basis begins with the definition of the various natural numbers. 1 is defined as the successor of 0, or briefly as 0'; 2 as 1', 3 as 2', and so on. By virtue of P2, this process can be continued indefinitely; because of P3 (in combination with P5), it never leads back to one of the numbers previously defined, and in view of P4, it does not lead back to 0 either.

5. Deduction of Addition using Peano's Axioms

As the next step, we can set up a definition of 'addition' which expresses in a precise form the idea that the addition of any natural number to some given number may be considered as a repeated addition of 1. The repeated addition of 1 is readily expressible using the successor relation. This definition of addition is considered as a first deduction and runs as follows:

D1: (a) $n + 0 = n$;
(b) $n + k' = (n + k)'$.

The two stipulations of this recursive definition completely determine the sum of any two integers. Consider, for example, the sum 1 + 2. According to the definitions of the numbers 2 and 1, we have

$$1 + 2 = 1 + 1' = 1 + (0')';$$

by D1 (b), $1 + (0')' = (1 + 0')' = ((1 + 0)')';$

but by D1 (a), and by the definitions of the numbers 2 and 3, $((1 + 0)')'=(1')'=2'=3$.

Consider, for example, the sum 3 + 2. According to the definitions of the numbers 2 and 1, we have

$$3 + 2 = 3 + 1' = 3 + (0')';$$

by D1 (b), $3 + (0')' = (3 + 0')' = ((3 + 0)')';$

but by D1 (a), and by the definitions of the numbers 4 and 5, $((3 + 0)')' = (3')' = 4' = 5$.

As above one can prove4 + 3 = 7 (left as an exercise). This proof also renders more explicit and precise the comments made earlier on the truth of the proposition that 4 + 3 = 7. Within Piano's system of arithmetic, its truth flows not merely from the definition of the concepts involved, but also from the postulates that govern these various concepts.

If we call the postulates and definitions of an axiomatic theory the "basics" concerning the concepts of that theory, then we may say now that the propositions of the arithmetic of the natural numbers are true by virtue of the basics which have been laid down initially for the arithmetical concepts. Note that our proof of the formula "1 + 2 = 3" and "3 + 2 = 5" repeatedly made use of the transitivity of identity. Latter it is accepted here as one of the rules of logic (Rules of inferences) which may be used in the proof of any arithmetical theorem. It is therefore, included among Peano's postulates no more than any other principle of logic.

6. Deduction of Multiplication using Peano's Axioms

Now, the multiplication of natural numbers may be defined using the following recursive definition. In a rigorous form the idea that a product *mn* of two integers *m, n* may be considered as the sum of *m* numbers each of which equals *n*.

D2: (a) $n.0 = 0$;
(b) $m.n' = m.n + m$.

Consider the example of "2.3 = 6". The proof of this is as follows:

We know that 2'=3. Therefore,

	2.3 = 2.(2)'= 2.2 + 2.	(By using D2, (b)).
Now,	2.2 = 2. 1' = 2.1+2.	(Again by using D2, (b)).
Further,	2.1 = 2.0' = 2.0 + 2 = 2	(By using both (a) and (b) of D2).

Thus finally we get;

$$2.3 = 2.2 + 2 = (2.1 + 2) + 2 = (2.1 + 2) + 2 = (2 + 2) + 2 = (4+2) = 6.$$

Note that 2 +2 = 4 and 4 + 2 = 6 can be deduced by using the deduction D1.

It now is possible to prove the familiar general laws governing addition and multiplication, such as the commutative, associative, and distributive laws. In precise for numbers *l, m, n* they are;

Commutative law: $m + n = n + m$ and $m.n = n.m$.

Associative law : $m + (n + l) = (m + n) + l$ and $(m.n).l = m.(n.l)$.

Distributive law : $m.(n + l) = (m.n) + (m.l)$ and $(m + n) . l = (m.l) + (n.l)$.

We illustrate the way of proving associative law and commutative law with respect to addition.

Associative Law: Let us denote the set of all whole numbers. Consider the set of numbers $S = \{z \in \mathrm{N} : (x+y)+z = x+(y+z), \forall x, y \in \mathrm{N}\}$. We first show that $1 \in S$. In fact, by using the notion of 'successor' we get $(x+y)+1 = (x+y)' = x+y' = x+(y+1)$. Thus $1 \in S$. Now to complete the proof we use P5. Assume $z \in S$, that is $(x+y)+z = x+(y+z)$ and using the assumption we need to show that z' is also in *S*.

$$(x+y)+z' = ((x+y)+z)' = (x+(y+z))' = x+(y+z)' = x+(y+z') \text{ -----(1)}$$

The above statement (1) follows by the repeated use of the definition of successor and our assumption $z \in S$. Thus from (1) we get $z' \in S$.

Commutative Law: First we show that $1+x = x+1$ for all $x \in \mathrm{N}$. Let $S_1 = \{x \in \mathrm{N} : 1+x = x+1\}$. Then by using D1, (a) and the definition of successor $1+0 = 1 = 0' = 0+1$, which implies $0 \in S_1$. Now assume $x \in S$. By using the definition of successor and by using the deduction D1, (a) and associative property, we get $1+x' = 1+(x+1) = (1+x)+1 = x'+1 = (x+1)+1 = x'+1$. Now by the postulate P5, it follows that $S_1 = \mathrm{N}$ and hence $1+x = x+1$ for all $x \in \mathrm{N}$------ (2).

Now consider $S_2 = \{y \in \mathrm{N} : y+x = x+y,\ x \in \mathrm{N}\}$. By the above discussion $0 \in S_2$. Assume $y \in S_2$. Now using associatively and (2) recursively we get;

$$\begin{aligned} y'+x &= (y+1)+x = y+(1+x) = y+(x+1) \\ &= (y+x)+1 = (x+y)+1 = x+(y+1) \\ &= x+y'. \end{aligned}$$

From postulate P5, the proof completes.

On the basis thus obtained, the various arithmetical and algebraic operations can be defined for the numbers of the new system, the concepts of function, of limit, of derivative and integral, and the familiar theorems relating to these concepts can be proved. So, finally, the huge system of mathematics as here delimited rests on the narrow basis of Peano's system.

> **"Every concept of mathematics can be defined using Peano's three primitives, and every proposition of mathematics can be deduced from the five postulates enriched by the definitions of the non-primitive terms."**

These deductions can be carried out, in most cases, by means of nothing more than the principles of formal logic.

Check Your Progress - 1

1. State Peano's axioms.
2. Illustrate deduction of addition by using Peano's axioms with an example.
3. Illustrate deduction of multiplication by using Peano's axioms with an example.
4. Derive commutative law with respect to addition of natural numbers.

1.3.3.2. The Nature of Mathematical Propositions -2

7. Mathematical Logic and Rules of Inferences

Logic is the basis of all mathematical reasoning and all automated reasoning. The rules of logic specify the meaning of mathematical statements. These rules help us understand and reason with statements. The rules of logic give a precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments. Apart from its importance in understanding mathematical reasoning, logic has numerous applications in Computer Science, varying from the design of digital circuits, to the construction of computer programs and verification of correctness of programs.

To understand mathematics, we must understand what makes up a correct mathematical argument, that is, a proof. Once we prove a mathematical statement is true, we call it a theorem. A collection of theorems on a topic organize what we know about this topic. To learn a mathematical topic, a person needs to actively construct mathematical arguments on this topic, and not just read exposition. Moreover, knowing the proof of a theorem often makes it possible to modify the result to fit new situations.

Everyone knows that proofs are important throughout mathematics, but many people find it surprising how important proofs are in computer science. In fact, proofs are used to verify that computer programs produce the correct output for all possible input values, to show that algorithms always produce the correct result, to establish the security of a system, and to create artificial intelligence. Furthermore, automated reasoning systems have been created to allow computers to construct their proofs.

8. Basic building Blocks of Logic

Statement (or Proposition): A Statement (or Proposition) is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

Examples: 1. Bangalore is the capital city of Karnataka.
2. 4 is a prime number.
3. $2 + 3 = 5$.

Statements (1) and (3) are True, whereas the proposition (3) is false. The following statements are examples of the statements which are not statements (or propositions).

1) How are you?
2) Please do a favor to me
3) $3 + x = 8$.

Note: In the later part of this unit we use the word *'proposition'* sometimes in place of '*statement'*. It is quite different from the proposition which we stated in the previous unit and used at the beginning of this unit.

We use letters to denote propositional variables (or statement variables), that is, variables that represent propositions, just as letters are used to denote numerical variables. The conventional letters used for propositional variables are *p,q,r,s,...* . If the truth value of a proposition is true, denoted by T, and the truth value of a proposition is false, denoted by F. The area of logic that deals with propositions is called the propositional calculus or propositional logic. It was first developed systematically by the Greek philosopher Aristotle more than 2300 years ago.

We now turn our attention to methods for producing new propositions from those that we already have. These methods were discussed by the English mathematician **George Boole in 1854** in his book The *Laws of Thought*. Many mathematical statements are constructed by combining one or more propositions. New propositions, called compound propositions, are formed from existing propositions using ***logical operators***.

(A) **Negation**: Let p be a proposition. The negation of p, denoted by $\neg p$ (also denoted by $\bar{p}$), is the statement "It is not the case that p."
The proposition $\neg p$ is read "*not p*". The truth value of the negation of p, is the opposite of the truth value of p.

Example: 1)p: Bangalore is the capital city of Karnataka.
$\neg p$ **:** Bangalore is **not the** capital city of Karnataka.
2) p: 4 is a prime number.
$\neg p$ **:** 4 **is not a** prime number.

The negation of a proposition can also be considered the result of the operation of the negation operator on a proposition. The negation operator constructs a new proposition from a single existing proposition. We will now introduce the logical operators that are used to form new propositions from two or more existing propositions. These logical operators are also called **connectives**.

(B) **Conjunction:** Let p and q be propositions. The conjunction of p and q, denoted by $p\wedge q$, is the proposition "p and q". The conjunction $p\wedge q$ is true when both p and q are true and is false otherwise.

Example: 1)p: Bangalore is the capital city of Karnataka. q: 4 is a prime number.
$p\wedge q$ **:** Bangalore is the capital city of Karnataka and 4 is a prime number.

Note: In logic, the word "but" sometimes is used instead of "and" in a conjunction. For example, the statement "The sun is shining, but it is raining" is another way of saying "The sun is shining and it is raining." (In natural language, there is a subtle difference in meaning between "and" and "but"; we will not be concerned with this nuance here.)

(C) **Disjunction:** Let p and q be propositions. The conjunction of *p*or*q*, denoted by $p\vee q$, is the proposition "*p*or*q*". The conjunction $p\vee q$ is true when *p*or*q*or both are true and is false otherwise.

Example: 1)p:Bangalore is the capital city of Karnataka. q: 4 is a prime number.

$p \vee q$ **:** Bangalore is the capital city of Karnataka or 4 is a prime number.

Note: The use of the connective or in a disjunction corresponds to one of the two ways the word or is used in English, namely, as an inclusive or. A disjunction is true when at least one of the two propositions is true.

(D) **Conditional Statement:** Let p and q be propositions. The conditional statement $p \rightarrow q$ is the proposition "if p, then q". The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise. In the conditional statement$p \rightarrow q$, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence).

The statement $p \rightarrow q$ is called a conditional statement because $p \rightarrow q$ asserts that q is trueon the condition that p holds. A conditional statement is also called an implication.

Note: Because conditional statements play such an essential role in mathematical reasoning, a variety of terminology is used to express $p \rightarrow q$. You will encounter most if not all of thefollowing ways to express this conditional statement:

"if p, then q""	"p implies q"
"if p, q""	"p only if q"
"p is sufficient for q "	"a sufficient condition for p is q"
"q if p""	"q whenever p"
"q when p""	"q is necessary for p"
"a necessary condition for p is q ""	"q follows from p"
"q unless $\neg p$ "	

Example: Let p be the statement "Maria learns discrete mathematics" and q the statement "Maria will find a good job". Then the statement $p \rightarrow q$ is "Maria will find a good job when she learns discrete mathematics.". the same also may be stated as follows :

"Maria will find a good job when she learns discrete mathematics."

or

"Maria will find a good job unless she does not learn discrete mathematics."

(E) **Biconditional Statement:** Let p and q be propositions. The *Biconditional* statement $p \leftrightarrow q$ is the proposition "p if and only if q". The Biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called bi-implications.

Example: The statement $p \leftrightarrow q$ is true when boththe conditional statements $p \rightarrow q$and $q \rightarrow p$ are true and is false otherwise. That is why we usethe words "if and only if " to express this logical connective and why it is symbolically writtenby combining the symbols $\rightarrow$ and $\leftarrow$. There are some other common ways to express $p \leftrightarrow q$:

"p is necessary and sufficient for q "
"if p then q, and conversely"
"p if q."

The last way of expressing the biconditional statement $p \leftrightarrow q$ uses the abbreviation "iff" for "if and only if." Note that $p \leftrightarrow q$hasexactlythesame truth value as$(p \rightarrow q) \wedge (q \rightarrow p)$.

Note: One should be aware that biconditionals are notalways explicit in natural language. In particular, the "if and only if" construction used in biconditionals is rarely used in common language. Instead, biconditionals are often expressed using an "if, then" or an "only if " construction. The other part of the "if and only if " is implicit. That is, the converse is implied, but not stated. For example, consider the statement in English "If you finish your meal, then you can have dessert." What is really meant is "You can have dessert if and only if you finish your meal." This last statement is logically equivalent to the two statements "If you finish your meal, then you can have dessert" and "You can have dessert only if you finish your meal." Because of this imprecision in natural language, we need to assume whether a conditional statement in natural language implicitly includes its converse. Because precision is essential in mathematics and in logic, we will always distinguish between the conditional statement $p \rightarrow q$ and the biconditional statement $\leftrightarrow q$.

Tautology and Contradiction: A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a ***tautology***. A compound proposition that is always false is called a ***contradiction***. A compound proposition that is neither a tautology nor a contradiction is called a ***contingency***.

9. Translating English Sentences to Logical Statements

Statements in mathematics and the sciences and in natural language often are imprecise or ambiguous. To make such statements precise, they can be translated into the language of logic. For example, logic is used in the specification of software and hardware, because these specifications need to be precise before development begins. Furthermore, propositional logic and its rules can be used to design computer circuits, to construct computer programs, to verify the correctness of programs, and to build expert systems. Logic can be used to analyze and solve many familiar puzzles. Software systems based on the rules of logic have been developed for constructing some, but not all, types of proofs automatically. We will discuss some of these applications of propositional logic in this section.

There are many reasons to translate English sentences into expressions involving propositional variables and logical connectives. In particular, English (and every other human language) is often ambiguous. Translating sentences into compound statements (and other types of logical expressions) removes the ambiguity. Note that this may involve making a set of reasonable assumptions based on the intended meaning of the sentence. Moreover, once we have translated sentences from English into logical expressions we can analyze these logical expressions to determine their truth values, we can manipulate them, and we can use rules of inference (which are discussed in the next section) to reason about them.

To illustrate the process of translating an English sentence into a logical expression, consider Examples 1 and 2.

Example 1: How can this English sentence be translated into a logical expression?

"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

Solution: There are many ways to translate this sentence into a logical expression. Although it is possible to represent the sentence by a single propositional variable, such as p, this would not be useful when analyzing its meaning or reasoning with it. Instead, we will use propositional variables to represent each sentence part and determine the appropriate logical connectives between them. In particular, we let a , c, and f represent "You can access the Internet from campus,""You are a computer science major," and "You are a freshman," respectively. Noting that "only if" is one way a conditional statement can be expressed, this sentence can be represented as$a \rightarrow (c \vee \neg f)$.

Example 2: How can this English sentence be translated into a logical expression?

"You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16years old."

Solution: Let q ,r , and s represent "You can ride the roller coaster," "You are under feet tall, "and "You are older than 16 years old," respectively. Then the sentence can be translated to $(r \wedge \neg s) \rightarrow \neg q$.

Of course, there are other ways to represent the original sentence as a logical expression, but the one we have used should meet our needs.

10. Logical Equivalence and Laws of Logic

The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology.The notation $p \equiv q$ denotes that p and q are logically equivalent.

Note: The symbol ≡ is not a logical connective, and $p \equiv q$ is not a compound propositionbut rather is the statement that $p \leftrightarrow q$ is a tautology. The symbol ⇔ is sometimes used insteadof ≡ to denote logical equivalence.

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity Laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination Laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent Laws

$\neg(\neg p) \equiv p$	Law of Double Negation
$p \lor q \equiv q \lor p$ $p \land q \equiv q \land p$	Commutative Laws
$(p \lor q) \lor r \equiv p \lor (q \lor r)$ $(p \land q) \land r \equiv p \land (q \land r)$	Associative Laws
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive Laws
$\neg(p \land q) \equiv \neg p \lor \neg q$ $\neg(p \lor q) \equiv \neg p \land \neg q$	DeMorgan's Laws
$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$	Absorption Laws
$p \lor \neg p \equiv T$ $p \land \neg p \equiv F$	Negation Laws

Table 1.3.1: Logical Equivalences

Example: Use De Morgan's laws to express the negations of "Miguel has a cell phone and he has a laptop computer" and "Heather will go to the concert or Steve will go to the concert."

Solution: Let p be "Miguel has a cellphone" and q be "Miguel has a laptop computer." Then"Miguel has a cellphone and he has a laptop computer" can be represented by $p \land q$. By thefirst of De Morgan's laws, $\neg(p \land q)$ is equivalent to $\neg p \lor \neg q$. Consequently, we can expressthe negation of our original statement as "Miguel does not have a cellphone or he does not havea laptop computer."

Let r be "Heather will go to the concert" and s be "Steve will go to the concert." Then "Heather will go to the concert or Steve will go to the concert" can be represented by$r \lor s$.By the second of De Morgan's laws, $\neg(r \lor s)$ is equivalent to $\neg r \land \neg s$. Consequently, we canexpress the negation of our original statement as "Heather will not go to the concert and Steve will not go to the concert."

11. Rules of Inferences

An ***argument*** in propositional logic is a sequence of propositions. All but the final proposition in the argument are called ***premises*** and the final proposition is called the ***conclusion***. An argument is valid if the truth of all its premises implies that the conclusion is true.

An ***argument form*** in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is valid no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

We can always use a truth table to show that an argument form is valid. We do this by showing that whenever the premises are true, the conclusion must also be true. However, this can be tedious approach. For example, when an argument form involves 10 a different propositional variable, to use a truth table to show this argument form is valid requires 2^{10} = 1024 different rows. Fortunately, we do not have to resort to truth tables. Instead, we can first establish the validity of some relatively simple argument forms, called rules of inference. These rules of inference can be used as building blocks to construct more complicated valid argument forms. We will now introduce the most important rules of inference in propositional logic.

The tautology $(p \wedge (p \rightarrow q)) \rightarrow q$ is the basis of the rule of inference called modus ponens, or the law of detachment. (Modus ponens is Latin for mode that affirms.) This tautology leads to the following valid argument form, which we have already seen in our initial discussion about arguments (where, as before, the symbol ∴ denotes "therefore"):

$$\begin{array}{c} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

Using this notation, the hypotheses are written in a column, followed by a horizontal bar, followedby a line that begins with the therefore symbol and ends with the conclusion.

Example1: Suppose that the conditional statement "If it snows today, then we will go skiing" and its hypothesis, "It is snowing today," are true. Then, by modus ponens, it follows that the conclusion of the conditional statement, "We will go skiing," is true.

There are many useful rules of inference for propositional logic. Perhaps the most widely used of these are listed in Table 1.3.2. We now give examples of arguments that use these rules of inference. In each argument, we first use propositional variables to express the propositions in the argument. We then show that the resulting argument form isa rule of inference from Table 1.3.2.

Rule of Inference	**Tautology**	**Name**
$\begin{array}{c} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus ponens
$\begin{array}{c} \neg q \\ \underline{p \rightarrow q} \\ \therefore \neg p \end{array}$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$	Modus tollens
$\begin{array}{c} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore p \rightarrow r \end{array}$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$\begin{array}{c} p \vee q \\ \underline{\neg p} \\ \therefore q \end{array}$	$((p \vee q) \wedge \neg p) \rightarrow q$	Disjunctive syllogism
$\begin{array}{c} \underline{p} \\ \therefore p \vee q \end{array}$	$p \rightarrow (p \vee q)$	Addition

$p \wedge q$ $\therefore p$	$(p \wedge q) \rightarrow p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$	Resolution

Table 1.3.2: Rules of Inferences

Example 2: Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

Solution: Let p be the proposition "It is sunny this afternoon," q the proposition "It is colderthan yesterday," r the proposition "We will go swimming," s the proposition "We will take a canoe trip," and t the proposition "We will be home by sunset." Then the premises become$\neg p \wedge q, r \rightarrow p, \neg r \rightarrow s,$ and$s \rightarrow t$.The conclusion is simply t. We need to give a valid argument with premises $\neg p \wedge q, r \rightarrow p, \neg r \rightarrow s,$ and $s \rightarrow t$ and conclusion t. We construct an argument to show that our premises lead to the desired conclusion as follows.

Steps	Reason
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

Example 3: Show that the premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."

Solution: Let p be the proposition "You send me an e-mail message," q the proposition "I willfinish writing the program," r the proposition "I will go to sleep early," and s the proposition "Iwill wake up feeling refreshed." Then the premises are $p \rightarrow q$, $\neg p \rightarrow r$, and $r \rightarrow s$. The desiredconclusion is $\neg q \rightarrow s$. We need to give a valid argument with premises $p \rightarrow q, \neg p \rightarrow r,$ and$r \rightarrow s$andconclusion$\neg q \rightarrow s$.

This argument form shows that the premises lead to the desired conclusion.

Steps	Reason
1. $p \to q$	Premise
2. $\neg q \to \neg p$	Contra positive of (1)
3. $\neg p \to r$	Premise
4. $\neg q \to r$	Hypothetical syllogism using (2) and (3)
5. $r \to s$	Premise
6. $\neg q \to s$	Hypothetical syllogism using (4) and (5)

Check in Progress -2

1. State different logical connectives and illustrate the use of each with an example.
2. List Laws of logic.
3. What is a Tautology and what is a contradiction. Give examples for each.
4. List all commonly used rules of inferences.

1.3.4. Let us Summarise

A ***proposition*** is the basic building block of logic. It is defined as a declarative sentence that is either 'True' or 'False', but not both. The first question arises is that; "How to decide the truth value of a proposition?" As argued so far that the validity of mathematics rests neither on its alleged self-evidential character nor on any experiential basis, but derives from the conditions which determine the meaning of the mathematical concepts, and that the propositions of mathematics are therefore essentially "true by definition."For the rigorous development of a mathematical theory proceeds not simply from a set of definitions but rather from a set of non-definitional propositions which are not proved within the theory; these are the postulates or axioms of the theory.

Once the primitive terms and the postulates have been laid down, the entire theory is completely determined. It is necessary also to specify the principles of logic which are to be used in the proof of the propositions, i.e., in their deduction from the postulates. These principles can be stated quite explicitly. They fall into two groups: *Primitive sentences, or Postulates* of logic, and *Rules of Deduction or Inferences.*

In this regard as a foundation, the basic concept of logic, logical connectives, logical equivalence, converting an ordinary English sentence into a logical statement have been discussed with illustrations. Translating sentences into compound statements removes the ambiguity. Moreover, once we have translated sentences from English into logical expressions, we can analyze these logical expressions to determine their truth values, we can manipulate them, and we can use rules of inference, which are also discussed in detail.

1.3.5. Answers to 'Check Your Progress - 1, and2'

Check Your Progress - 1

1. Write P1 to P5 (ref section 1.3.3.3.)
2. Refer deduction of 2+3=5 by using Peano's axioms.
3. Refer deduction of 2.3=6 by using Peano's axioms.
4. Refer commutative law. [First, prove $x + 1 = 1 + x$, then assume $x + y = y + x$ and show that $x + y' = y' + x$.]

Check in Progress - 2

1. Logical connectives are "NOT", 'OR', 'AND', 'IF, THEN', and 'IF AND ONLY IF' For examples refer to Basic building blocks of logic in Section 1.3.3.6.
2. To List Laws of logic refer the Table 1.3.1.
3. Tautology is a logical statement, which is always true and a contradiction is a logical statement, which is always false. If p and q are nay two logical statements, then $p \rightarrow (p \vee q)$ is a tautology and $p \wedge \neg p$ is a tautology.
4. List of rules of inferences is given in Table 1.3.2.

1.3.6. Unit end Exercises

1. Describe the analytic character of mathematical propositions.
2. Using Peano's axioms derive the addition '2+2=4' and the multiplication '2.2=6'.
3. Prove associative law and commutative law of multiplication using Peano's axioms.
4. Translate the given statement into propositional logic using the propositions provided.
   (a) You cannot edit a protected Wikipedia entry unless you are an administrator. Express your answer in terms of p: "You can edit a protected Wikipedia entry" and q:"You are an administrator."
   (b) You can see the movie only if you are over 18 years old or you have the permission of a parent. Express your answer in terms of p: "You can see the movie," q: "You are over 18 years old," and p: "You have the permission of a parent."
5. Use rules of inference to show that the hypotheses "Randy works hard," "If Randy works hard, then he is a dull boy," and "If Randy is a dull boy, then he will not get the job" imply the conclusion "Randy will not get the job."
6. Use rules of inference to show that the hypotheses "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on," "If the sailing race is held, then the trophy will be awarded," and "The trophy was not awarded" imply the conclusion "It rained."

1.3.7. References

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3 Kenneth H Rosen, "Discrete Mathematics and its Applications', McGraw-Hill, 7th Edition 2007.

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Block 1 : Nature, Aims and Objectives of Mathematics

Unit 4 : Need for Establishing General Objectives for Teaching Mathematics

Unit Structure

1.4.1. Learning Objectives

After completing this Unit, the student teachers will be able to

- Explain the meaning of Aims;
- Explain the meaning of Objectives;
- Explain the difference between Aims and Objectives;
- Explain the use of Objectives in Teaching and Testing;
- Explain the meaning of General Instructional Objectives;
- Explain the meaning of Specific Outcome of Learning; and
- Explain the need for establishing general objectives.

1.4.2. Introduction

Education is a process of bringing about changes in the individual in desired directions, such as the development of interests, attitudes and skills, to carry out certain activities. This helps the child to lead a happy productive and socially acceptable life. An objective presents the endpoint towards which action is directed and therefore it reflects the purposefulness of the educational process. It represents the first step in the teaching and learning process because it is the starting point of activities, planning and instruction. It also provides the basis for the selection of evaluation procedures and curriculum development. Thus objectives validate whole teaching-learning process. In this unit, we shall see the meaning of Aim and Objectives and establish the need for General Objectives of teaching Mathematics.

1.4.3. Learning Points and Learning Activities

1.4.3.1. Meaning and nature of Aims and Objectives

Exercise

Recall the different phases of your life and write in the space provided what was the biggest professional dream of your life. How did you work towards achieving it? (Write your dream even if it has not been achieved).

Dream of your life	Steps you took in achieving your dream

As you recalled your dream you must have realized that you had a goal in life which you which you wanted to fulfill. To achieve this goal I am sure you have taken various steps which would help you in achieving your goal. For example, if your goal was to become a teacher, you might have selected subjects in your college that would help you take up a teacher training course. Further, you have joined the course which can make you a teacher.

Similarly in education teaching-learning requires having a goal to achieve something that is necessary for life. We shall therefore see how to set this goal and also plan the outcome in this unit.

A. Meaning of Aims

Aims are general and long term goals and may be common to more than one subject. Long-term goals refer to high-level aims and tend to be related to broad reasons, why a particular subject or activities are being organized or why a particular course is being taught. Thus aims or long term goals can be regarded as expressions of strategy. Objectives are specific, immediate and expressions of strategy. While objectives are specific, immediate and attainable goals, specific to one subject, precise and clearly defined, objectives are more directly concerned with what specifically is being attempted over a relatively short period.

B. Meaning of Objectives

An instructional objective is a statement of the expected result. It is a description of the learning outcome that the teacher hopes will result from the instruction, whether in a lesson unit or course. It is a statement of what students should be able to do at the end of the learning period that they could not do beforehand. Thus the term *'Objective'* may be defined as

> "An objective is a point or end view of the possible achievement in terms of what a student can do when the whole educational system is directed towards educational aims."

Thus, an objective is a part of the aim which a school can hope to achieve. Hence an objective is a narrower term when compared to an aim. It is a statement that describes what the pupil will do or be able to do towards the realization of an educational aim. When a pupil attains an objective he realizes a part of the broad aim. In other words, an objective is a statement of the terminal behavior expected of the pupils at the conclusion of a period of learning.

In other words, we can say that the objective is a statement or a form of category which suggests any kind of change. It indicates the direction of the pupil's growth and provides the basis for the selection of evaluation procedures. Objectives provide a link between teachers, pupils, testers and parents by focusing their attention on intended outcomes of learning. Thus objectives validate the process of education. Hence objectives have the following characteristics.

- They provide direction to the activities.
- They help for the planned change.
- They provide a basis for organizing teaching-learning activities.

The objectives are classified into two categories

i. **Educational Objectives:** Educational Objectives are broad and philosophical in nature. They are related to the schools and educational systems. E.J Frust has well defined "Educational objective as a desired change in the behaviour of a person that we try to bring about through education". According to B.S. Bloom " Educational objectives are not only the goals towards which the curriculum is shaped and towards which the instruction is guided, but they are also the goals which provide the detailed specification of the curriculum and use of evaluation techniques" The educational objectives are achieved with the help of teaching or instructional objectives.

ii. **Teaching Objectives:** Teaching objectives are narrow and psychological in nature. Teaching objectives may be achieved during a certain period in the classroom. These are related to the expected change in behavior of the child. So they are also called behavioral objectives. Teaching objectives are directly related to the learning process and they are well defined, definite, clear, specific and measurable. These give direction to the learning processes, learning-experiences and teaching. They provide the foundation of the entire educational structure. Therefore, teaching objectives are also called instructional objectives. The teaching strategies methods and techniques are selected based on teaching or instructional objectives.

C. Difference between Aims and Objectives

Aims and the objectives may be compared based on the following points

Aims	**Objectives**
1. Aims are very broad and comprehensive	1. Objectives are narrower and specific
2. Philosophy and sociology is the main source of aim	2. Psychology is the main source of objectives.
3. They are not definite and clear	3. They are definite and clear
4. They are difficult to achieve	4. They can be achieved conveniently
5. Long duration is required for the aims to be achieved	5. Achieved within a short duration i.e. within the classroom period
6. They are subjective	6. They are objective
7. These cannot be evaluated	7. These can be evaluated
8. These include objectives	8. Objectives are a part of the aims
9. They are related with the whole education system and whole curriculum	9. These are related with the teaching of any specific topic
10. It is the responsibility of the school, society and nation to achieve them	10. Generally teacher is responsible
11. These are theoretical and indirect	11. Objectives are direct and concerned with the teaching learning process.
12. Aims are formal	12. These are functional and informative.

Check Your Progress - 1
Give two differences between Aims and Objectives

1.4.3.2. Need for Establishing Objectives

Exercise II
Answer the following questions.

Do you need goal in life? If so, explain why?

As you answered the above question, you surely have realized that goals are very essential in life. They serve as the guiding light to where and how you will move forward in life. If your goal is clear the path you have to travel becomes easy and clearly visible. Even teaching and learning goals are very essential to achieve the desired outcome. We state and classify these goals under Aims and Objectives. Now let us see why these objectives are needed for our teaching learning process.

A. Use of Objectives in Teaching and Testing

Instruction involves three distinct kinds of activities: teaching, learning and evaluation. If the instructional process is to be effective, all three activities must be oriented to certain common objectives. The objectives form the pivot of the entire teaching-learning process. Objectives should be stated for each course, unit and topic. They are the mental skills that students should develop as a result of teaching. The objectives direct the pupils as to what he or she is expected to do, what should be the minimum level of acceptance for his or her performance and under what conditions it will be achieved. After objectives have been stated the teacher plans the teaching and learning experiences needed for the students to attain the objectives. Learning experiences are those activities that are planned with a specific purpose of bringing about the desired changes in the behavior of the students. Realization of the objectives, to a great extent, depends upon careful selection and planning of appropriate learning experiences. These experiences may include seminars, laboratory work, discussions, audiovisual presentations, research paper writing, projects etc. These techniques provide active student's involvement in learning and different techniques work for different learning needs. True learning is not merely the acquisition of knowledge or certain skills; it is a change in behavior brought about by training or experience. The changes the learner experiences are the direct outcome of his interaction with the learning environment. The teacher then selects an appropriate evaluation technique to assess the student's performances in terms of whether or not the students have attained the objectives described. This evaluation helps the teacher in ascertaining.

a. Whether or not the desired changes have taken place in the pupil's behavior.
b. The attainability and feasibility of the stated objectives.
c. The effectiveness of the learning experiences provided.

If the teacher is not satisfied with the outcome of the evaluation, the teacher has to critically appraise all the three components of the teaching namely, objectives, learning experiences and evaluation techniques and accordingly change or modify any of these three components.

B. General Instructional Objectives (GIO) and Specific Outcomes of Learning (SOL)
General Instructional Objectives (GIO)

The General Instructional Objectives are for a course and can apply to any item of the curriculum/syllabus. They are intended to assist in defining and carrying out broad educational aims. By specifically stating the kind of outcome of student learning desired, these objectives can be used to clarify teaching methods, learning experiences and materials needed for particular content and course.

Example:

- The pupil acquires knowledge of mathematical terms, facts, concepts, principles, theorems etc.
- The pupils understand the meaning of mathematical terms, facts etc.
- The pupil applies mathematical principles to new and unfamiliar situations.

These objectives are also known as non-behavioral objectives as they do not depict the overt behaviour of the student. The statement of such objectives contains verbs like "knows", "understands" etc. The pupil understands the polynomial division or the pupil knows the place value of binary numbers are examples for non-behavioral objectives as 'understanding' by the student or 'acquisition of knowledge' by the student cannot be observed by the teacher and therefore they are not observable behaviors.

Specific Outcomes of Learning (SOL)

Specific Outcomes of Learning consists of statements defining the specific performances, which we adopt as evidence that a student has actually reached his objective, all of which are precise and measurable. These objectives are also known as behavioral objectives as the statements of these objectives contain an action verb that displays an overt behaviour of the learner. Pupil explains the polynomial division or the pupil states the properties of binary addition are behavioral objectives, an explanation by the student or stating by the student are observable behaviors. There will be a large number of these objectives from which a sample may be taken as an indication that the student has attained the broader General Instructional Objectives.

Example:

- The pupils recall definitions of mathematical terms or concepts
- The pupil recognizes mathematical symbols
- The pupil lists properties of geometrical figures
- The pupil classifies geometrical figures
- The pupil gives reason for mathematical statements
- The pupil establishes the relationship among mathematical concepts.
- The pupil formulates a hypothesis for solving a given problem
- The pupil selects principles relevant to the problem presented.

C. Criteria for Judging Instructional Objectives

In choosing general instructional objectives, it is helpful to have criteria against which to judge whether or not the objectives are relevant and useful.

- Attainability – Within the realm of possibility.
- Validity – In line with the aims of education
- Comprehensiveness – Covering all the behaviors and content material
- Precision – Clear and unambiguous
- Feasibility – For application
- Appropriateness – For yielding specific outcomes
- Reasonable in number
- Consistent with one another.

D. Need for establishing General Objectives for teaching Mathematics

a) **Makes clear what a student should be able to do on completing the course:** General Objectives are stated for the collective outcome of a course and hence they make clear what a student is capable of doing at the end of that course. This includes the attitude toward mathematics, skills related to mathematics, operations and relations, values and on the whole the syllabus stipulated to that course.

b) **Helps in identifying the learning outcomes and specific outcomes of learning:**To identify the learning outcomes and express them as specific outcomes of learning one needs a framework. General Objectives provides this framework.

c) **Guides the teacher to select appropriate methods and approaches of teaching:** It is very essential to select the right methods and approaches to teaching mathematics if the learning has to be optimum. The general objectives guide the teacher in selecting them.

d) **Provides the frame of reference for decisions about selection and organization of the subject matter, the mode of instruction and techniques of evaluation:** General objectives make us know what is suitable for a particular course and what is expected of that course. Hence a teacher will have a perfect view of what and how to teach and evaluate.

e) **Directs pupils as to what he or she is expected to achieve by the end of instruction:** General Objectives also help the pupils to have an idea as to what is expected of them by the end of the course. This gives them a direction in which they have to proceed.

f) **Helps the teacher to select the appropriate evaluation technique:** Evaluation forms an important part of any course. And since the general objectives specify what is to be learnt during the course, it guides in selecting the right evaluation techniques which are necessary to evaluate the outcomes.

Check Your Progress - 2

1. Explain the importance of the use of Objectives in Teaching and Testing.
2. Ascertain the need for establishing the general Objectives of teaching Mathematics.

1.4.4. Let us Summarise

- **Meaning of Aims:** Aims are general and long term goals and may be common to more than one subject. Long-term goals refer to high-level aims and tend to be related to broad reasons, why a particular subject or activities are being organized or why a particular course is being taught.
- **Meaning of Objectives:** An instructional objective is a statement of the expected result. It is a description of the learning outcome that the teacher hopes will result from the instruction, whether in a lesson unit or course.
- The objectives are classified into two categories

Educational Objectives: Educational Objectives are broad and philosophical in nature.
Teaching Objectives: Teaching objectives are narrow and psychological in nature.

- **Difference between Aims and Objectives**
  a) Aims are very broad and comprehensive while objectives are narrower and specific.
  b) Philosophy and sociology is the main source of aim while psychology is the main source of objectives.
- **Use of Objectives in Teaching and Testing:** Instruction involves three distinct kinds of activities: teaching, learning and evaluation. If the instructional process is to be effective, all three activities must be oriented to certain common objectives.
- **General Instructional Objectives (GIO):** The General Instructional Objectives are for a course and can apply to any item of the curriculum/syllabus. They are intended to assist in defining and carrying out broad educational aims.
- **Specific Outcomes of Learning (SOL):** Specific Outcomes of Learning consists of statements defining the specific performances, which we adopt as evidence that a student has actually reached his objective, all of which are precise and measurable.
- Need for establishing the general objectives for teaching Mathematics
  a) Makes clear what a student should be able to do after a period of instruction
  b) Helps in identifying the learning outcome and specific outcomes of learning
  c) Guides the teacher to select appropriate methods and approaches of teaching
  d) Provides the frame of reference for decisions about selection and organization of the subject matter, the mode of instruction and techniques of evaluation.
  e) Directs pupils as to what he or she is expected to achieve by the end of instruction.
  f) Helps the teacher to select the appropriate evaluation technique.

1.4.5. Answers to 'Check Your Progress - 1 and 2'

Check Your Progress - 1

1 Aims are very broad and comprehensive while objectives are narrower and specific.
2 Philosophy and sociology is the main source of aim while psychology is the main source of objectives

Check Your Progress - 2

1. The objectives form the pivot of the entire teaching-learning process. Objectives should be stated for each course, unit and topic. They are the mental skills that students should develop as a result of teaching. The teacher then selects an appropriate evaluation technique to assess the student's performances in terms of whether or not the students have attained the objectives described. This evaluation helps the teacher in ascertaining.
   a. Whether or not the desired changes have taken place in the pupil's behavior.
   b. The attainability and feasibility of the stated objectives.
   c. The effectiveness of the learning experiences provided.

If the teacher is not satisfied with the outcome of the evaluation, the teacher has to critically appraise all the three components of the teaching namely, objectives, learning experiences and evaluation techniques and accordingly change or modify any of these three components.

2. Need for establishing the general objectives for teaching Mathematics.
   a. Makes clear what a student should be able to do after a period of instruction
   b. Helps in identifying the learning outcome and specific outcomes of learning
   c. Guides the teacher to select appropriate methods and approaches of teaching
   d. Provides the frame of reference for decisions about selection and organization of the subject matter, the mode of instruction and techniques of evaluation.
   e. Directs pupils as to what he or she is expected to achieve by the end of instruction.
   f. Helps the teacher to select the appropriate evaluation technique.

1.4.6. Unit end Exercises

1. Explain the meaning of Aims in teaching Mathematics
2. Explain the meaning of Objectives.
3. What are the two types of Objectives? Explain
4. Differentiate between Aims and Objectives.
5. Explain the uses of Objectives in teaching and testing.
6. What are the General Objectives of Instruction? Explain with examples
7. What are the Specific Outcomes of Learning? Explain with examples.
8. Specify the criteria for Judging Instructional Objectives.
9. Elucidate the need for General Objectives in Mathematics.

1.4.7. References

1. Dr. Anice James (2019), Teaching of Mathematics. Neelkamal Publication Pvt. Ltd.
2. J.C. Agarwal, Essentials of Examination System, Vikas Publishing House Pvt. Ltd.
3. Dr. A.K. Kulshreshtha (2015) Teaching of Mathematics, Lal Book Depot Publications
4. Dr. S.K. Mangal, Pedagogy of Mathematics, Tandon Publications, Ludhiana.

Block 1 : Nature, Aims and Objectives of Mathematics

Unit 5 : Aims and General Objectives of Teaching Mathematics

Unit Structure

1.5.1. Learning Objectives

After completing this Unit, the student teachers will be able to

- Explain the Aims of teaching Mathematics;
- Explain the Bloom's Taxonomy of teaching Objectives;
- State the objectives of Teaching Mathematics as per National Policy of Education (1986); and
- State the Objectives of Teaching Mathematics as per New Curriculum Document (2000).

1.5.2 Introduction

We all know that aims and objectives form the foundation of any subject. Well stated aim and a well-stated objective can give clarity to the path, an educator has to follow to give the best learning experiences to the pupils. Aims give a general direction to where the subject must lead while the objectives give specific pictures of the outcome of learning. Several policies and committees have worked on giving the best guidelines in the formation of Aims and Objectives related to different subjects. Mathematics being one of the important subjects requires ample attention in setting up its Aims and Objectives so that it is utilized in the best possible way in the intellectual progress of mankind. In this Unit, we shall see the different Aims of Mathematics and Objectives of Mathematics as stated by the National Policy of Education and the New Curriculum Document.

1.5.3. Learning Points and Learning Activities

1.5.3.1. Aims of Teaching Mathematics

EXERCISE I

What are your intentions while teaching a particular topic in Mathematics? (Write all that you think you want to achieve by teaching Mathematics in general)

I surely know that while teaching any subject for that matter one looks at how useful the subject is to the pupils. And while answering the above question you would surely have pointed out this fact of utility. Apart from it, a subject should also have scope for the personality development of the pupils. Now we shall go through the Aims of teaching

Mathematics and see what are the different aspects that are intended to be developed while learning this subject.

Aims of teaching mathematics can be classified under the following heads

a) Utilitarian or Practical Aim

The following are the practical aims of teaching mathematics.

- To enable the students to have clear ideas about the number concept.
- To give the individual an understanding of ideas and operations in number and quality needed in daily life.
- To enable the individual to have a clear comprehension of the way the number is applied to all measures but most particularly to those frequently used concepts such as length volume, area, weight, temperature, speed etc.
- To enable the individual to become proficient in the four fundamental operations of addition, subtraction, multiplication and division.
- To provide the basis of mathematical skills and processes, that are needed for vocational purposes.
- To enable the learner to acquire and develop mathematical skills and attitudes to meet the demands of (i) daily life (ii) future mathematical work and (iii) work in the related fields of knowledge.
- To enable the students to make appropriate approximations.
- To enable the learner to understand the concept of ratio and scale drawing, read and interpret graphs, diagrams and tables.
- To enable the individual to apply his mathematics to a wide range of problems that occurs in daily life.

b) Disciplinary Aim

The teaching of Mathematics intends to realize the following disciplinary aims.

- To provide opportunities that enable the learners to exercise and discipline mental faculties.
- To help the learner in the intelligent use of reasoning power.
- To develop constructive imagination and inventive faculties.
- To develop the character through systematic and orderly habits.
- To help the learner to be original and creative in thinking.
- To help the individual to become self-reliant and independent.

c) Cultural Aim

The cultural aim can be summarized as follows:

- To enable the learner to appreciate the part played by mathematics in the culture of the post and that it continues to play in the present world.
- To enable the student to appreciate the role played by mathematics in preserving and transmitting our cultural traditions.
- To enable him to appreciate various cultural arts like drawing, design making, painting, poetry, music, sculpture and architecture.
- To provide through mathematical ideas, aesthetic and intellectual enjoyment and satisfaction and to allow creative expression.
- To help the students explore creative fields such as art and architecture.
- To make the learner aware of the strengths and virtues of the culture he has inherited.
- To develop in the individual an aesthetic awareness of mathematical shapes and patterns in nature as well as the products of our civilization.

d) Social Aims

The important social aims of teaching mathematics are as under

- To develop in the individual and awareness of the mathematical principles and operations which will enable the individual to understand and participate in the general social and economic life of his community.
- To enable the student to understand how the methods of mathematics such as scientific, intuitive, deductive and inventive are used to investigate, interpret and to make the decision in human affairs.
- To help the pupil acquire social and moral values to lead a fruitful life in society.
- To help the pupil in the formation of social laws and social order needed for social harmony.
- To provide the pupils scientific and technological knowledge necessary for adjusting to the rapidly changing society and social life.
- To help the learner appreciate how mathematics contributes to his understanding of the natural phenomenon.
- To help the pupil interpret social and economic phenomenon.

Check Your Progress - 1

List the different Aims of teaching Mathematics.

1.5.3.2. Objectives of Teaching Mathematics

Exercise II

Now that you are familiar with the idea of Objectives, list a few specific changes in behaviour that you expect from the pupils, after learning the Mathematics subject.

As we all know Objectives point out at the specific changes in behaviour or the specific learning experiences that a pupil will have during a learning process. Now let us go through the objectives of teaching mathematics.

Dr. Benjamin S. Bloom (1956) has classified the changes of behaviour into three categories or Domains

1. Cognitive Domain
2. Affective Domain
3. Psychomotor Domain

Dr. B.S. Bloom and his associates at the University of Chicago gave the classification of objectives of all three domains.

1. Classification of cognitive domain or objectives by Bloom (1956)
2. Affective domain by Krathwohl (1964)
3. Psychomotor Domain by Simpson (1986)

Dr. Bloom concentrated on the study of the cognitive domain. He assumed that in thinking about a problem a hierarchy of cognitive process is involved. While teaching, a teacher follows this hierarchical order. This classification of objectives is known as 'Taxonomy of Educational Objectives" or "Blooms Taxonomy" of objectives.

Taxonomy of Teaching Objectives

Serial No.	Cognitive Domain Category	Affective Domain Category	Psychomotor Domain Category
	Dr. B. S. Bloom (1956)	**Krathwohl (1964)**	**Simpson(1969)**
1.	Knowledge	Receiving	Impulsion
2.	Comprehension	Responding	Manipulation
3.	Application	Valuing	Control
4.	Analysis	Conceptualization	Co-ordination
5.	Synthesis	Organization	Naturalization
6.	Evaluation	Characterization	Habit Formation

Cognitive Objectives: Cognitive Objectives stress that the pupils should acquire more and more knowledge. It was defined to include all those activities which deal with the recall or recognition of knowledge and the development of intellectual abilities and skills.

Affective Objective: Affective objective is concerned with the attitude, interest, emotions, values and mental tendencies of the pupils. This part of the taxonomy also includes appreciations and social adjustment of the child.

Psychomotor Objective: This third part of taxonomy and includes the manipulative and motor-skill areas. The physical actions involved in handwriting, playing, using equipments, making an outline, drawing figures and many others are in the psycho-motor domain.

Objectives of Teaching Mathematics – National Policy of Education (1986)
At the end of the high school stage, a pupil should be able to –

- Acquire knowledge and understanding of the terms, concepts, principles, processes, symbols and mastery of computational and other fundamental processes that are required in daily like and for higher learning in mathematics.
- Develop skills of drawing, measuring, estimating and demonstrating.
- Apply mathematical knowledge and skills to solve problems that occur in daily life as well as problems related to higher learning in mathematics or allied areas.
- Develop the ability to think, reason, analyze and articulate logically.
- Appreciate the power and beauty of mathematics.
- Show interest in mathematics by participating in mathematical competitions, and engaging in its learning, etc.
- Develop reverence and respect towards great mathematicians, particularly towards great Indian mathematicians for their contributions to the field of mathematical knowledge.
- Develop necessary skills to work with modern technological devices such as calculations, computers, etc.

Objectives of Teaching Mathematics – New Curriculum Document (2000)
The learners-

- Consolidate the mathematical knowledge and skills acquired at the upper primary stage.
- Acquire knowledge and understanding of the terms, symbols, concepts, principles, process, proofs, etc.
- Develop mastery of basic algebraic skills.
- Develop drawing skills.

- Apply mathematical knowledge and skills to solve real mathematical problems by developing abilities to analyze, to see interrelationships involved, to think and reason.
- Develop the ability to articulate logically.
- Develop awareness of the need for national unity, national integration, protection of the environment, observance of small family, norms, removal of social barriers, and elimination of sex biases.
- Develop the necessary skills to work with modern technological devices such as calculators, computers, etc.
- Develop interest in mathematics and participate in mathematical competitions and other mathematical club activities in the school.
- Develop appreciation for mathematics as a problem-solving tool in various fields for its beautiful structures and patterns, etc.
- Develop reverence and respect towards great mathematicians, particularly towards the Indian mathematicians for their contributions to the field of mathematics.

Objectives of Teaching Mathematics

The objectives of teaching mathematics at the secondary stage may be classified as under:

- Knowledge and Understanding objectives
- Skill objectives
- Application objectives
- Attitude objectives
- Appreciation and Interest objectives

A. Knowledge and Understanding Objectives

The student acquires knowledge and understanding of:

- Language of mathematics i.e., the language of its technical terms, symbols, statements, formulae, definitions, logic, etc.
- Various concepts i.e., the concept of number, the concept of direction, concept measurement.
- Mathematical Ideas, like facts, principles, processes and relationships.
- The development of the subject over the centuries and contributions mathematicians.
- Inter-relationship between different branches and topics of mathematics etc.
- The nature of the subject of mathematics.

B. Skill Objectives

The subject helps the student to develop the following skills:

- He acquires and develops skills in the use and understanding of mathematical language.
- He acquires and develops speed, neatness, accuracy, brevity and precision in mathematical calculations.
- Learns and develops the technique of problem-solving.
- Develops and ability to estimate, check and verify results.
- Develops and ability to perform calculations orally and mentally.
- Develops and ability to think correctly, to draw conclusions, generalizations and inferences.
- Develops skills to use mathematical tools, and apparatus.
- Develops essential skills in drawing geometrical figures.
- Develops skills in drawing, reading, interpreting graphs and statistical tables.
- Develops skills in measuring, weighing and surveying.
- Develops skill in the use of mathematical tables and ready references.

C. Application Objectives

The subject helps the student to apply the above-mentioned knowledge and skills in the following way:

- Able to solve mathematical problems independently.
- Makes use of mathematical concepts and processes in everyday life.
- Develops ability to analyze, to draw inferences, and to generalize from the collected data and evidence.
- Can think and express precisely, exactly, and systematically by making proper use of mathematical language.
- Develops the ability to use mathematical knowledge in the learning of other subjects especially sciences.
- Develops the students' ability to apply mathematical in his future vocational life.

D. Attitude Objectives

The subject helps to develop the following attitudes:

- The student learns to analyze the problems.
- Develops the habit of systematic thinking and objective reasoning.
- Develops heuristic attitude and tries to discover solutions and proofs with his independent efforts.
- Tries to collect enough evidence for drawing inferences, conclusions and generalizations.
- Recognizes the adequacy or inadequacy of given data in relation to any problem.
- Verifies his results.
- Understands and appreciates logical, critical and independent thinking in others.
- Expresses his opinions precisely, accurately, logically and objectively without any biases and prejudices.
- Develops self-confidence for solving mathematical problems.
- Develops personal qualities namely, regularity, honesty, objectivity, neatness and truthfulness.
- Develops mathematical perspective and outlook for observing the realm of nature and society.

E. Appreciation and Interest Objectives

The student is helped in the acquisition of appreciation and interest in the following way:

- Appreciates the role of mathematics in everyday life.
- Appreciates the role of mathematics in understanding his environment.
- Appreciates mathematics as the science of all sciences and art of all arts.
- Appreciates the contribution made by mathematics in the development of civilization and culture.
- Appreciates the contribution of mathematics to field and other branches.
- Develops the interest in the learning of the subject.
- Feels enter by mathematical recreations.
- Develops interest in the activities of mathematics clues.
- Develops interest in the active library reading, mathematical projector.
- Appreciates the aesthetic nature of mathematics by observing symmetry, similarity, order and arrangement in mathematical facts, principles and processes.
- Appreciates the contribution of mathematics in the development of other branches of knowledge.
- Appreciates the recreational values of the subject and learn to utilize it in his leisure time.

- Appreciates the vocational value of mathematics.
- Appreciates the role of mathematical language, graphs and tables in giving precision and accuracy to his expression.
- Appreciates the power of computation developed through the subject.
- Appreciates the role of mathematics in developing his power of acquiring knowledge.
- Appreciates mathematical problems, their intricacies and difficulties.
- He develops interest in the learning of the subject.
- He feels entertained by mathematical recreations.
- He takes an active interest in the activities of the mathematics club.
- He takes an active interest in the active library reading, mathematical projects, and doing practical work in the mathematics laboratory.

Check Your Progress - 2
What are the classifications of Objectives of teaching Mathematics at Secondary Stage?

1.5.4. Let us Summarise

Aims of teaching mathematics can be Classified under the heads – (a) Utilitarian or Practical Aim, (b) Disciplinary Aim, (c) Cultural Aim, (d) Social Aims.

Dr. B. S. Bloom (1956) has classified the changes of behaviour into three categories or Domains as – (1) Cognitive Domain, (2) Affective Domain and (3) Psychomotor Domain.

Taxonomy of Teaching Objectives

Sl. No.	Cognitive Domain Category	Affective Domain Category	Psychomotor Domain Category
	Dr. B. S. Bloom (1956)	**Krathwohl (1964)**	**Simpson (1969)**
1	Knowledge	Receiving	Impulsion
2	Comprehension	Responding	Manipulation
3	Application	Valuing	Control
4	Analysis	Conceptualization	Co-ordination
5	Synthesis	Organization	Naturalization
6	Evaluation	Characterization	Habit Formation

Classification of Objectives of Teaching Mathematics

- ✓ Knowledge and Understanding objectives
- ✓ Skill objectives
- ✓ Application objectives
- ✓ Attitude objectives
- ✓ Appreciation and Interest objectives

1.5.5. Answers to 'Check Your Progress - 1 and 2'

Check Your Progress - 1

1. **Aims of teaching mathematics**
   a) Utilitarian or Practical Aim
   b) Disciplinary Aim
   c) Cultural Aim
   d) Social Aims

Check Your Progress - 2

1. Classification of Objectives of Teaching Mathematics

- ✓ Knowledge and Understanding objectives
- ✓ Skill objectives
- ✓ Application objectives
- ✓ Attitude objectives
- ✓ Appreciation and Interest objectives

1.5.6 Unit end Exercises

1. Which are the different Aims of teaching Mathematics? Explain
2. Explain the Bloom's Taxonomy of teaching Objectives
3. State the objectives of Teaching Mathematics as per the National Policy of Education (1986)
4. State the Objectives of Teaching Mathematics as per New Curriculum Document (2000)
5. What are the different classifications of Objectives of teaching Mathematics at Secondary Stage?
6. State the Knowledge and Understanding objectives for teaching Mathematics at Secondary Stage.
7. State the Skill objectives for teaching Mathematics at Secondary Stage.
8. State the Application objectives for teaching Mathematics at Secondary Stage.
9. State the Attitude objectives for teaching Mathematics at Secondary Stage.
10. State the Appreciation and Interest objectives for teaching Mathematics at Secondary Stage.

1.5.7 References

1 Dr. Anice James (2019), Teaching of Mathematics. Neelkamal Publication Pvt. Ltd.

2 J.C. Agarwal, Essentials of Examination System, Vikas Publishing House Pvt. Ltd.

3 Dr. A.K. Kulshreshtha (2015), Teaching of Mathematics, Lal Book Depot Publications

4 Dr. S.K. Mangal, Pedagogy of Mathematics, Tandon Publications, Ludhiana.

5 www.pedagogybyvasu.blogspot.com.

Block 1 : Nature, Aims and Objectives of Mathematics

Unit 6: Writing Learning Objectives and Teaching Points of Various Content Areas in Mathematics like Algebra, Geometry, Trigonometry etc.

Unit Structure

1.6.1. Learning Objectives

After completing this Unit, the student teachers will be able to

- Explain the meaning of Learning Objective;
- Explain the process of writing Learning Objective;
- Write Instructional Objectives in Mathematics;
- Write Learning Objectives in Mathematics; and
- Write the Objective for teaching Arithmetic, Algebra, Geometry and Trigonometry.

1.6.2. Introduction

Now that have understood what are the Aims and Objectives of teaching Mathematics it is necessary to look at it in a little more detailed manner. The details of objectives lie in how clearly it states what the outcome of learning is. As the major goal of every teaching activity is the optimum level of learning, one needs to be clear about what does that optimum level consist of. This clarity can be found only when the objectives are stated with a focus on learning. Such objectives which has its focus on learning and which states learning in its behavioral forms comes to be called learning objectives. In this unit, we shall discuss how to write learning objectives in Mathematics.

1.6.3. Learning Points and Learning Activities

1.6.3.1. Learning Objectives of Mathematics

Exercise I

Now that you have learned the concept of Objectives, Write the Objectives for teaching the topic "Circles".

You would surely have written a list of objectives for teaching the topic "Circle". But have you been able to specify in your objectives those specific behavioral changes that you would expect to see in your pupils at the end of their learning the topic? Writing the learning objectives in specific terms is a skill in itself. In the present unit, we shall discuss it.

A. Writing Learning Objectives

A Learning Objective is an objective that describes what students should know or be able to do at the end of the course. In other words, the objectives state the specific outcomes of learning that a student achieves when an Instructional Objective has been achieved.

Hence for each General Instructional Objective, it is necessary to write Specific Outcomes of Learning that will state the precise behaviour or performance that is expected of a student. Each general instructional objective can have many specific outcomes of learning under it. These are smaller units of performance and can be precisely measured by tests of various kinds.

There are five elements which when used in writing a Specific Outcome of Learning give the clearest definition for student performance that can be used for both teaching and testing. The five elements are as follows

a) Performer (The student, the trainer, the learner, etc.)
b) Action Required (An action verb, Example: identifies, compares, describes, distinguishes, analyses, classifies, etc.)
c) Task (Include a task to be performed. Example: Compares the properties, explains the derivation.)
d) Conditions (Include any condition that may be required. Example: Compares the properties of the given triangles.)
e) Criteria for judgement (Any relevant criteria for clarity. Example: Explains the phenomena with at least two examples, computers with speed and accuracy.)

Three Qualities to be Maintained-

In writing an SOL three qualities must be maintained, if SOL is to serve the purpose of communication between teacher, pupil and examiner.

- Use clear, precise action verbs.
- Must be feasible in terms of student's level, nature of the content and learning experiences.
- Must be observable and measurable.

B. Instructional Objective in Mathematics

I. Remembering: It is cognitive level.

The pupil –

1. Recalls the mathematical laws, principle, rule formulae, etc.
2. Recognizes the mathematical laws, principle, formulae, etc.

II. Understanding: Goes deep into the content

The pupil –

- Cites or gives examples.
- Gives reasons.
- Identifies.
- Compares.
- Finds relationship.
- Based on observation draw conclusions.
- Draws inference or the result.
- Converts verbal form to symbolic form or vice versa.
- Classifies mathematical data.

III. Applying:

The Pupil –

- Analyses the problem into its components.
- Judges the adequacy of the given data.
- Suggests the alternate methods.
- Suggest the most appropriate method.
- Generalizes.

IV. Skill:
The Pupil –

- Reads mathematical figures, statements, problems, charts, tables, etc.
- Labels the geometrical figure.
- Draws the most appropriate, neat and proportionate geometrical figures.
- Solves oral problems quickly and accurately.
- Solves written problems quickly and accurately.

C. Learning Objectives in Mathematics

1. Remembering: The pupil acquires knowledge of mathematics.

Learning outcomes: The pupil -

- Recalls mathematical terms, facts, processes, principles, formulae definitions, signs and symbols, relationships, generalizations etc.,
- Recognizes terms, instruments, process, formulae, signs and symbols, relationships, generalisations etc.,

2. Understanding: The pupil develops an understanding of mathematics.
Learning Outcomes: The pupil -

- Explains mathematical terms, concepts, principles, etc., in his own words.
- Defines mathematical terms and concepts.
- State mathematical principles, relationships etc.
- Gives illustrations for mathematical concepts, principles, etc.,
- Identifies mathematical terms, concepts, relationships, figures, processes etc.
- Finds similarities between mathematical terms, concepts, relationships, figures etc.
- Differentiates between mathematical terms, concepts, relationships, figures etc.
- Classifies mathematical terms, concepts, figures etc.
- Verbalises symbolic relationships and vice versa.
- Frames mathematical formulae, generalisations based on data.
- Uses the formula to solve problems.
- Substitute relevant numbers, symbols and signs in the mathematical formulae and operations.
- Calculate the answers for given problems.
- Uses appropriate units to write answers.
- Finds solutions for given problems.

3. Applying: The pupil applies knowledge of mathematics to novel situations.

Learning outcomes: The pupil -

- Analyses a problem or data into component parts.
- Judges the adequacy, inadequacy or superfluity of data.
- Establishes relationships among data.
- Gives a number of methods of solving a problem.
- Select the most appropriate formulae or principles or methods or processes to solve problem.
- Reasons deductively.
- Reasons inductively.
- Makes a generalization.

- Draws inferences.
- Predicts results based on data.

4. Skill:

a) The pupil acquires skills in handling mathematical instruments with ease.

Learning Outcomes: The pupil -

- Draws freely satisfactory free-hand figures.
- Selects the most appropriate mathematical instruments.
- Takes necessary precautions in taking measurements while constructing geometrical figures.
- Takes measurements correctly.

b) Drawing geometrical figures and Graphs

Learning Outcomes: The pupil -

- Draws figures to given specifications.
- Draws figures quickly.
- Uses appropriate marking to denote different parts of a figure.

c) Computation:

Learning Outcomes: The pupil -

- Does oral calculation correctly.
- Does oral calculation quickly.
- Does a written calculation correctly.
- Does a written calculation quickly.
- Uses correct notations and symbols.
- Avoids unnecessary steps in the solution of a problem.
- Is systematic in working on a problem.

d) Reading of Tables, Charts, Graphics, etc.

Learning Outcomes: The pupil -

- Selects appropriate mathematical tables.
- Uses mathematical tables, charts, ready reckoners etc., correctly.
- Co-ordinates the different sections of the graphs correctly.
- Reads a graph correctly.

Check Your Progress - 1

1. What are Learning Objectives?
2. Select a topic of your choice in Mathematics and write the Learning Objectives for it.

1.6.3.2. Objectives of Teaching Arithmetic, Algebra, Geometry and Trigonometry

A. Objectives of Teaching Arithmetic

a) Understanding Arithmetic for selecting Teaching Points

Arithmetic is a branch of mathematics that consists of the study of numbers especially the properties of the traditional operations on them- addition, subtraction, multiplication and division. Arithmetic is an elementary part of number theory and number theory is considered to be one of the top-level divisions of modern mathematics along with algebra, geometry and analysis. The terms arithmetic and higher arithmetic were used until the beginning of the 20^{th}

century as synonyms for number theory and are sometimes still used to refer to a wider part of number theory.

Arithmetic is the science of numbers and the art of computation. Historically arithmetic was developed out of a need for a system of counting. It is considered to be essential for efficient and successful living. That is why arithmetic is divided as the science that deals with numbers with relations between numbers, numbers in term, or abstraction arising from such concrete situations as counting measuring and ordering the various quantities and objects that we encounter in everyday life. The teaching of arithmetic has to fulfill two responsibilities.

- The inculcation of an appreciative understanding of the number system and an intelligent proficiency in its fundamental process.
- The socialization of number experiences.

b) Objectives of teaching arithmetic

- To develop fundamental arithmetic concepts like the concept of number, order, units of measurement, size, shape etc among the pupils.
- To train the pupils in mathematical thinking i.e. to understand the statement to analyze it and to arrive at the right conclusions.
- To arouse pupil's interest in the quantitative side of the world around them and its use as a simple tool in business.
- To develop accuracy and facilitate the simple computation of the fundamental process among pupils.
- To develop speed and accuracy in arithmetical calculation and computation among pupils
- To impart to pupils a working knowledge of practical arithmetical applications, which are useful in life.
- To develop a sense of appreciation among the pupils for the use of arithmetic in daily life.
- To prepare pupils to learn other branches of mathematics and also pursue higher studies in mathematics.

B. Objectives of Teaching Algebra

a) Understanding Algebra for selecting Teaching Points

Algebra in its most general form is the study of mathematical symbols and the rules for manipulating these symbols. It is a unifying thread of almost all of mathematics. It includes everything from elementary equation solving to the study of abstractions such as groups, rings and fields. Algebra is called the science of letter. It refers to the methods of reasoning about numbers by employing letters to represent their relationship. Algebra is concerned largely with the structure of the number system, operations with numbers and statements involving numbers as well as the solution of problems. Algebra is a language used to develop and express much of the scientific data. Algebra comprehended a more general treatment of numbers and number relations than this arithmetic. It is concerned with the general statement about numerical situations. Algebra refers to the operation of taking a quantity from one side of the equation to another by changing its signs. It presents a radically new and different approach to the study of quantitative relationships characterized by a new symbolism, new concepts, and a new language much higher degree of generalization and abstraction than has been encountered in arithmetic. But it is primarily taught for manipulative skills. Solutions of problems by equations give power of generalization and use of formulae and idea of functionality.

b) Objectives of teaching Algebra -

- To develop among pupils the skill of identifying patterns
- To develop the skill of representing real-world situations using expressions of equations.
- To develop the skill of simplifying expressions using order of operations.
- To develop among pupils the skill of solving multi-step equations.
- To develop the skill of converting equations to graph.
- To provide an effective way of expressing complicated relations.
- To inculcate the power of analysis.
- To verify the results more simply and satisfactorily.
- To provide a new and refined approach in the study of abstract mathematical relationships through the use of new symbolism.

C. Objectives of Teaching Geometry:

a) Understanding Geometry for selecting Teaching Points

The word geometry originally means the measurement of earth. It is the science of lines and figures it is the science of space and extent. It deals with the position, space and size of bodies but nothing to do with their material properties. Geometry has two important aspects –
Demonstrative Geometry – It deals with the shape, size and position of figures by pure reasoning based on definitions, self-evident truths and assumptions. Euclid, a great Greek Mathematician was the father of demonstrative Geometry. His methods are intuitional, observational, intentional, constructive, informal, creative, and experimental and so on.
Practical Geometry – It covers the constructional work of the subject. Most of the work is directly or indirectly based on demonstrative Geometry.

b) Objective of teaching Geometry

- To enable the learner to acquire the knowledge of geometrical facts.
- To implement geometrical principles like equality, symmetry similarity in every nature of things.
- To develop the ability to draw accurate figures.
- To demonstrate the nature and the power of pure reason.
- To systematize the information received by the pupils in the pre-school stage.
- To aid the pupils in becoming familiar with the basic geometrical concepts and space perceptions and in understanding the fundamental techniques such as the use of set square, protractor, compass, etc.
- To acquaint the pupil with the geometrical notations.

D. Objectives of Teaching Trigonometry

a) Understanding Geometry for selecting Teaching Points

Trigonometry is a branch of mathematics that studies relationships between side lengths and angles of triangles. Trigonometry is found throughout geometry, as every straight-sided shape may be broken into a collection of triangles. Further still trigonometry has astoundingly intricate relationships to other branches of mathematics, in particular complex numbers, infinite series, logarithms and calculus. The word trigonometry is a 16th century Latin derivative from the Greek words for triangle (trigonon) and measure (metron).

b) Objectives of Teaching Trigonometry

The students will be able,

- To understand trigonometric ratios and identities.

- Find the value of trigonometric ratios of some specific angles.
- Determine the trigonometric ratios of complementary angle.
- Apply the trigonometric identities in proving the given statement.
- To apply the knowledge of trigonometry to solve daily life problems.
- To find heights and distances.
- To appreciate the use of trigonometry to solve problems.
- To develop creative thinking and reasoning.
- To appreciate its usefulness in technology
- To understand the relationship between trigonometry and other branches of mathematics.

Check Your Progress - 2
Enumerate the objectives of teaching Arithmetic, Algebra, Geometry and Trigonometry

1.6.4. Let us Summarise

- **Writing Learning Objectives**
  A Learning Objective is an objective that describes what students should know or be able to do at the end of the course. In other words, the objectives state the specific outcomes of learning that a student achieves when an Instructional Objective has been achieved.

- **Instructional Objective in Mathematics**
  - ✓ **Remembering:** Recalls and Recognizes.
  - ✓ **Understanding**: Cites or gives examples, Gives reasons, Identifies, Compares, Finds relationship, based on observation draws conclusion, Draws inference or the result, Converts verbal form to symbolic form or vice versa, Classifies mathematical data.
  - ✓ **Applying:** Analyses the problem into its components, Judges the adequacy of the given data, suggests the alternate methods, Suggest the most appropriate method, Generalizes
  - ✓ **Skill:** Reads mathematical figures, statements, problems, charts, tables, etc., Labels the geometrical figure. Draws the most appropriate, neat and proportionate geometrical figures, Solves oral problems quickly and accurately, Solves written problems quickly and accurately.
  - **Arithmetic:** Arithmetic is a branch of mathematics that consists of the study of numbers especially the properties of the traditional operations on them- addition, subtraction, multiplication and division.
  - **Algebra:** Algebra in its most general form is the study of mathematical symbols and the rules for manipulating these symbols. It is a unifying thread of almost all of mathematics. It includes everything from elementary equation solving to the study of abstractions such as groups, rings and fields.
  - **Geometry:** The word geometry originally means measurement of earth. It is the science of lines and figures it is the science of space and extent. It deals with the position, space and size of bodies but nothing to do with their material properties.
  - **Trigonometry:** Trigonometry is a branch of mathematics that studies relationships between side lengths and angles of triangles.

1.6.5. Answers to 'Check Your Progress - 1 and 2'

Check Your Progress - 1

1. A Learning Objective is an objective that describes what students should know or be able to do at the end of the course. In other words, the objectives state the specific outcomes of learning that a student achieves when an Instructional Objective has been achieved.
2. Pupils write objectives for the topic of their choice.

Check Your Progress - 2

Refer section 1.6.3.2.

1.6.6. Unit end Exercises

1. Explain the things to be kept in mind while writing the Learning Objectives in Mathematics.
2. Write the Instructional Objectives for teaching Mathematics.
3. Write the Learning Objective for teaching Mathematics.
4. Write the Objectives for teaching Arithmetic.
5. Write the Objectives for teaching Algebra
6. Write the Objectives for teaching Geometry
7. Write the Objectives for teaching Trigonometry

1.6.7. References

1. Dr. Anice James (2019), Teaching of Mathematics. Neelkamal Publication Pvt. Ltd.
2. J.C. Agarwal, Essentials of Examination System, Vikas Publishing House Pvt. Ltd.
3. Dr. A .K. Kulshreshtha (2015) Teaching of Mathematics, Lal Book Depot Publications.
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5. www.pedagogybyvasu.blogspot.com.
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Block 2 : School Mathematics Curriculum and Instruction

Unit 1 : Objectives of Curriculum

Unit Structure

2.1.1. Learning Objectives

After completing this Unit, the student teachers will be able to

- Explain the meaning of Curriculum;
- State the definitions of Curriculum;
- Explain the place of mathematics in the School Curriculum;
- Explain the reasons for keeping Mathematics in School Curriculum;
- Explain the need for Mathematics Curriculum; and
- State the aims and objectives of Mathematics Curriculum.

2.1.2. Introduction

A pupil's learning during his/her schooling is determined by the curriculum. The content which the pupil is going to learn, the method which the pupil will follow to learn, the experience which the pupil will have during this period is all dependent on the curriculum that is followed by his/her school. Hence the curriculum finds a very important place in the teaching-learning process. As almost the curriculum forms the core of the teaching-learning process, its planning and designing have to be done meticulously to cater to every type of pupil and give them the best learning experiences. To plan the best Mathematics Curriculum one has to be aware of its aims and objectives and the need for it. In this unit, we shall discuss the meaning, need, aims and objectives of Mathematics Curriculum.

2.1.3. Learning Points and Learning Activities

2.1.3.1. Meaning of Curriculum

Exercise I

In the columns provided below list the different subjects (both curricular and non-curricular) you studied during your school and in the place provided explain the different activities your performed and different things you learnt under this subject.

Subject	Activities Performed	Aspects Learnt
Eg: Craft	**Flower making, Stitching baby dresses, Doll making**	**Different techniques of doll making and flower making, Sewing**

In the above columns, I am sure that you must have filled in all your the subjects you studied in school like English, Kannada etc and recalled all those activities you performed under each subject and the learning that happened. If you observe them carefully you see that there is an intentional plan that was underway to give you all those experiences. Each of the learning and the experiences that you receive during school together can be called the Curriculum. Now let us try to understand the meaning of curriculum and look at the definitions given by different people.

Meaning of Curriculum

The word curriculum is derived from the Latin word "currere" which means 'run' or 'race course'. Thus curriculum means a cause to be run for reaching a certain goal. Indeed the curriculum is like a racecourse for the pupil. As a person runs to win the race, in the same way a pupil undergoes various experiences to run through the curriculum to teach the educational goals. Therefore, etiologically it is clear the curriculum is that path or way over which a child runs to achieve the aims of education. Hence in the wider sense curriculum dignifies all those activities and learning experiences which a child undergoes in and outside the class according to his needs attitudes and interests. But in the narrow sense or according to the old concept, the meaning of curriculum is supposed to be a list of reading material. Reading material was generally called a study subject. In modern times, definitions of curriculum have become widespread.

Definition of Curriculum

The curriculum has been defined differently by many authors and over the years the focus is being shifted from 'course of study' to 'learning activities and experiences'.

Alberty A. and Alberty E (1959): Curriculum is the sum total of student activities which the school sponsors to achieve its objectives.

H. Robert Beck and W. Walter Cook: The sum of the educational experiences that children have in school

Cunnigham: Curriculum is a tool in the hands of the artist (teacher) to mould his material (the pupil) per his ideals in his studio (the school).

Darek Rowntree in A dictionary of Education (1981): The total structure of ideas and activities developed by an educational institution to meet the needs of the students and to achieve educational aims.

Thus it is clear that curriculum covers not only the course of study but also covers all the wider areas f individual and group life. It also encompasses all the meaningful and desirable activities outside the school, provided that these and planned, organized and used educationally. A good curriculum is the sum total of good learning experiences that the pupils have to achieve the goals of education which determine the direction of these experiences.

Definitions of Curriculum giving different Dimensions and View Points

Descriptive: Those aspects of schooling which have been deliberately planned comprise the curriculum.

Perspective: Curriculum is a set of content units which are arranged in a way that the learning of each unit may be accomplished as a single act provided the capabilities described by specified prior units in the sequence have already been mastered by the learner.

Static: Curriculum is a judiciously organized subject matter.

Dynamic: It is an organized set of processes, procedures, programmes and the likes which are applied to learners To achieve certain kinds of objectives.

Scientific: Curriculum is purely and simply a teaching strategy. A teaching strategy is, in turn, conceived of as being a series of goal-oriented activities or procedures to be carried out by teachers with respect to a class of learners, and in the context of a syllabus or a body of subject matter.

Check Your Progress - 1

1 What is the etymological meaning of Curriculum?
2 Define Curriculum.

2.1.3.2 Mathematics Curriculum

Exercise II

Recall your mathematics classes during your school days; did you feel you needed this subject? Recall and see if your answer was 'YES' or 'NO'. Whatever may be your answer give reasons for it.

Some of you all might have liked learning this subject or some of you all may not have liked it, yet I am sure you had your reasons for it. Now we shall give a thought to why mathematics is required to be learnt and see what is its place in the school curriculum.

A. Place of Mathematics in School Curriculum

Curriculum includes all those activities, experiences and environments which the child receives during his educational career under the guidance of educational authorities. The curriculum is the total education of the child.

Curriculum touches all the aspects of the life of the pupils – the need and interest of pupils environment which should be educationally congenial to them, ways and manners in which their interest can be kindled and warmed up, the procedures and approaches which cause effective learning among them, the social efficiency of the individual and how they fit in with the community around.

In education, the importance and the place of a particular subject depend on the fact that "to what extent the subject helps achieve the aims of education". If any subject is more useful for achieving educational objectives then its importance increases accordingly. Since ancient times mathematics has played a vital role in achieving the aims of education, as compared to others. The present age is the age of science and information. Whatever, technological and physical progress being made, shall be correspondent to the role of mathematics. Kothari Commission has explained about placing mathematics as a compulsory subject up to higher secondary or tenth standard and has said: "Mathematics should be made a compulsory subject for the students of 1^{st} to 10^{th} standard as a part of general education".

But some people lay more emphasis on making it an optional subject after the eighth standard and therefore various reasons were framed against this proposal.

1. It is a very difficult subject and its learning requires a sharp brain and intelligence, as many children will face difficulties in gaining knowledge.
2. It is only an imagination that mental abilities, discipline, culture, social and moral developments can be done by mathematics.
3. The number of failures in mathematics, in high school examination are more as compared to that of other subjects.
4. In higher studies, mathematical knowledge is important for those who keep their main subject as physics, chemistry or mathematics. It is useless for others.
5. Every student can't become an engineer or technician, then what is the necessity of mathematics for all.

In this way, the reason for forbidding the compulsion of mathematics up-to tenth standard seems so ideological.

All great educationalists like Herbert, Pestalozzi etc have accepted mathematics as a symbol of human development. Accepting mathematics as the best means of intellectual and cultural development, these educationists placed mathematics on the top of the curriculum. Thus we can give certain logical points regarding mathematics as a compulsory subject. These are as follows

1. If mathematics is not given an important place in the curriculum then students would not get an opportunity for mental training and in the absence of which their intellectual development might be affected.
2. For gaining the knowledge of mathematics no innate power is required, which is separate from the ability to study of another subject.
3. Training of reasoning, thinking, discipline, self-confidence and emotions are developed in students by mathematics.
4. Through mathematics, the child learns to gain knowledge systematically.
5. It is needed either forwardly or adversely for studying almost all the subjects because it is considered as the basis of science and every art.

Thus based on the above discussion, we can conclude that mathematics is the only subject whose knowledge is needed for the whole life. It can be possible only when every child will study mathematics as a compulsory subject up to the tenth standard. Mathematics occupies a prominent place in a person's life from an engineer to technician or laborer to finance minister and other businessmen, all need the help of mathematics according to their requirements. The knowledge of this subject is indispensable and is bound to grow as the need grows. A mathematical approach is essential for any progress. Any approach devoid of mathematical consideration is likely to lead to failure. If anybody wants to get success in his life, he must have recourse to mathematics.

B. Reasons for keeping Mathematics in School Curriculum

1. **Mathematics is the basis of all science:** The different branches of science likewise – physics, chemistry, astronomy, biology, medical science, geology, astrology etc are

the important subjects which are based on mathematics for eg. Area, volume, weight, density, number of atoms and electrons, medicines all are related to mathematical study.

2. **Mathematics is related to Human Life:** Right from getting up in the morning till going to bed we need the help of mathematics. For purchasing, planning our day, every aspect involves the use of mathematics. Today in the modern age, the knowledge of mathematics is essential and more important in one form or the other. Engineering, banking and other business which are directly linked with mathematics, for them mathematics works like a foundation brick and the business which are indirectly related to mathematics also depends totally on it. Besides these in our daily routine also we need general mathematical knowledge.
3. **Mathematics generates logical attitudes:** Mathematics gives training to different faculties of our mind. To solve a mathematical problem a child has to think logically. Every step is related to another step based on some logic with which the child develops his mental abilities and it further affects his intellectual development.
4. **Mathematics provides a definite way of thinking:** The children who study mathematics develop an attitude with which they learn to work systematically, regularly and properly. Along with this it also develops a logical thinking in them.
5. **Mathematics is an exact science:** By the study of mathematics, the child develops the attitude to accept the knowledge of mathematics in an exact form. All mathematical concepts, formulae, facts are related to exactness and thus it removes the feeling of doubt. For example 2+2=4 which cannot be 3 or 5 etc.
6. Mathematics provides an opportunity to develop the mental abilities of the child
7. Mathematics helps in character formation as well as morality.
8. It develops the discipline
9. The language of mathematics is universal
10. Knowledge of mathematics is useful in the study of other school subjects.
11. Mathematics deals with significant, abstracts and consistent structures.
12. Mathematics is the study of sets and structures
    a) **The algebraic structures –** In algebraic structures, we study operations of addition, multiplication and generalization.
    b) **The topological structures –** The topological structures include different concepts like limit, neighborhood or nearness etc
    c) **The order structure –** This type of structures include concepts like greater than and less than etc

C. Need for Mathematics Curriculum

The Mathematics curriculum seeks to answer the following

- What learning experiences are most appropriate for the attainment of the objectives of teaching mathematics?
- What kind of subject matter (syllabus) would provide the best learning experiences to the students to realize the objectives?
- What kind of mathematical skills are to be developed among the students to lead a fruitful life in the modern and technologically advanced society.
- What kind of resources is necessary for effective handling of instruction in the mathematics classroom?
- What contribution can the learning of mathematics make towards individual and national development?
- What kind of mathematical skills would help the students in taking up a fairly good number of vocations?

- What is the type of mathematical knowledge necessary for the study of other subjects and higher education in mathematics?
- What are the best methods of teaching and the most appropriate techniques of evaluation best suited to the student?

Thus the mathematics curriculum forms the basis for the entire mathematics education. It is the pivot on which the whole process of teaching-learning revolves. It provides the necessary insight to the mathematics teacher in the selection of the learning activities, teaching methods, learning resources and evaluation techniques. It helps the students in getting trained in skills necessary for his individual and social development, for selecting vocations in life, studying other subjects and pursuing higher education in mathematics. A good curriculum provides experiences that are best suited to the age of the learner, the emotional, physical and intellectual maturity of the learner and his previous experiences and learning.

Check Your Progress - 2
Give reasons as to why mathematics should be kept in the school curriculum.

2.1.3.3. Aims and Objectives of Mathematics Curriculum

Exercise III

Now that you know why mathematics needs to be a part of the school curriculum, reflect on your experience in mathematics and also know how to write objectives, write down the objectives that you think need to be a part of the mathematics curriculum.

I am sure you have stated good enough objectives that need to be a part of the Mathematics Curriculum. Now we shall arrange them in an orderly manner.

Narrow Aim

- The narrower aim of teaching Mathematics at school is to develop useful capabilities, particularly those relating to numeracy-numbers, number operations, measurements, decimal and percentage.

Broader Aim

- The broader aim is to develop the child to think and reason mathematically, to pursue assumptions to their logical conclusions and to handle abstractions. School Mathematics curriculum should help the children learn to enjoy Mathematics. How can these visions materialize? The vision discussed earlier needs to be put to practice in the form of goal-directed activities.

Specific Objectives

The following objectives will help us in realizing the vision of the school Mathematics curriculum:

- Attain proficiency in fundamental mathematical skills;
- Comprehend basic mathematical concepts;
- Develop desirable attitudes to think, reason, analyze and articulate logically;
- Acquire efficiency in sound mathematical applications within Mathematics and in other subject areas;
- Attain confidence in making intelligent and independent interpretations; and

- Appreciate the power and beauty of Mathematics for its application in science, social sciences, humanities, arts, etc.

Check Your Progress - 3

State the objectives of Mathematics Curriculum.

2.1.4. Let us Summarise

- **Meaning of Curriculum:** The word curriculum is derived from the Latin word "currere" which means 'run' or 'race course'. Thus curriculum means a cause to be run for reaching a certain goal.
- **Definition of Curriculum**
  Alberty A. and Alberty E (1959): Curriculum is the sum total of student activities which the school sponsors to achieve its objectives.
  H. Robert Beck and W. Walter Cook: The sum of the educational experiences that children have in school.
- **Place of Mathematics in School Curriculum:** In education, the importance and the place of a particular subject depends on the fact that "to what extent the subject helps achieve the aims of education". If any subject is more useful for achieving educational objectives then its importance increases accordingly. Since ancient times mathematics has played a vital role in achieving the aims of education, as compared to others.
- **Reason for keeping Mathematics in school curriculum**
  1. Mathematics is the basis of all science
  2. Mathematics is related to Human Life
  3. Mathematics generates logical attitudes
  4. Mathematics provides a definite way of thinking
  5. Mathematics is an exact science
  6. Mathematics provides an opportunity to develop the mental abilities of the child
  7. Mathematics helps in character formation as well as morality.
  8. It develops the characteristic of discipline
  9. The language of mathematics is universal
  10. Knowledge of mathematics is useful in the study of other school subjects.
  11. Mathematics deals with significant, abstracts and consistent structures.
  12. Mathematics is the study of sets and structures
      a) The algebraic structures
      b) The topological structures
      c) The order structure
- **Need for Mathematics Curriculum:** Mathematics curriculum forms the basis for the entire mathematics education. It is the pivot on which the whole process of teaching-learning revolves. It provides the necessary insight to the mathematics teacher in the selection of the learning activities, teaching methods, learning resources and evaluation techniques.
- **Objectives of Mathematics Curriculum**
  - ✓ Attain proficiency in fundamental mathematical skills;
  - ✓ Comprehend basic mathematical concepts;
  - ✓ Develop desirable attitudes to think, reason, analyze and articulate logically;
  - ✓ Acquire efficiency in sound mathematical applications within Mathematics and in other subject areas;
  - ✓ Attain confidence in making intelligent and independent interpretations; and
  - ✓ Appreciate the power and beauty of Mathematics for its application in science, social sciences, humanities, arts, etc.

2.1.5. Answers to 'Check Your Progress - 1, 2 and 3'

Check Your Progress - 1

1 The word curriculum is derived from the Latin word "currere" which means 'run' or 'race course'.
2 The sum of the educational experiences that children have in school.

Check Your Progress - 2

Reason for keeping Mathematics in school curriculum

- Mathematics is the basis of all science
- Mathematics is related to Human Life
- Mathematics generates logical attitudes
- Mathematics provides a definite way of thinking
- Mathematics is an exact science
- Mathematics provides an opportunity to develop mental abilities of the child
- Mathematics helps in character formation as well as morality.
- It develops the characteristic of discipline
- The language of mathematics is universal
- Knowledge of mathematics is useful in the study of other school subjects.
- Mathematics deals with significant, abstracts and consistent structures.

Check Your Progress - 3

Objectives of Mathematics Curriculum

- Attain proficiency in fundamental mathematical skills;
- Comprehend basic mathematical concepts;
- Develop desirable attitudes to think, reason, analyze and articulate logically;
- Acquire efficiency in sound mathematical applications within Mathematics and in other subject areas;
- Attain confidence in making intelligent and independent interpretations; and
- Appreciate the power and beauty of Mathematics for its application in science, social sciences, humanities, arts, etc.

2.1.6. Unit end Exercises

1. Discuss the meaning of Curriculum, stating appropriate definitions.
2. Discuss the place of mathematics in School Curriculum.
3. Justify the reasons for keeping Mathematics in School Curriculum.
4. Explain the need for Mathematics Curriculum.
5. State the aims and objectives of Mathematics Curriculum.

2.1.7. References

1. Dr. Anice James (2019), Teaching of Mathematics. Neelkamal Publication Pvt. Ltd.
2. J.C. Agarwal, Essentials of Examination System, Vikas Publishing House Pvt. Ltd.
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Block 2 : School Mathematics Curriculum and Instruction

Unit 2 : Principles for Designing Curriculum

Unit Structure

2.2.1 Learning Objectives

After completing this Unit, the student teachers will be able to

- State the principles of curriculum construction;
- Explain the principles of curriculum construction ;
- State the guidelines for the selection of topics for mathematics curriculum;
- Explain the organization of mathematics curriculum; and
- Explain the principles of curriculum organization.

2.2.2. Introduction

In our previous unit, we have understood what the curriculum is and what are the objectives of curriculum construction. Curriculum construction needs a strong basis for its development since it deals with all the experiences related to the subject that a pupil is going to have. One has to pay attention to the all-round development of the pupils and hence cater to the different aspects of development which include intellectual, psychological and physical.

Mathematics is a subject, where a pupil needs to develop various skills and this requires a meticulous plan to accommodate all the areas of mathematical skill development. And to achieve all this, the principle of curriculum construction needs to be thoroughly followed. In this unit, we shall look at the different principles of mathematics curriculum construction and also discuss as to how the mathematics curriculum needs to be organized in terms of its topics and contents in the Secondary School.

2.2.3. Learning Points and Learning Activities

2.2.3.1. Principles of Curriculum Construction and Selection of topics in Mathematics Curriculum

Exercise I

Select a topic of your choice, in Mathematics and write down why you need to study that topic and what are your expectations when you study that topic?

As you tried answering the above question you must have observed that you thought of the usefulness of that topic. Anything that we learn needs to be useful to us. This will happen only when the learning that is intended is suitable to our level, our environment and makes sense to us. To provide such learning experiences to the pupils, one has to take the help of previously researched and proven principles which can be the guiding light in curriculum construction. Hence we shall see what are the different principles of curriculum construction, for the construction of a good mathematics curriculum and also apply it in selecting and organizing the mathematics concepts for Secondary School Pupils.

A. Principles of Curriculum Construction

There are certain basic principles of curriculum planning which should form the basis for the construction of a good mathematics curriculum. They are as follows

Principle of Child-Centeredness: The curriculum shall be based on the present needs and capabilities of the children. The curriculum should help in developing initiative, co-operation and social responsibility among the children. This implies that the curriculum should meet the physical, intellectual, emotional and social needs of the pupils.

Curriculum should provide a fullness of experience for the students: The curriculum should help the children in living a wholesome and self-fulfilling life. The curriculum should be responsive to the fast-changing realities of life.

Curriculum should be dynamic and not static: Curriculum should reflect growth and movements of life. The curriculum should accommodate the latest developments in mathematics, science and information technology.

Curriculum should be related to everyday life: The curriculum should provide sufficient opportunities for the students to relate what they learn in classrooms with daily life experiences. The problems and theories that form a part of the mathematics curriculum should be real and should help in solving everyday life problems.

It must take into account the economic aspect of the life of the people to whom an educational institution belongs: The curriculum should provide adequate opportunities for the children to become economically self-reliant through participation in SUPW (Socially Useful Productive Work) and vocationalisation. Moreover, the curriculum should prepare the child in taking up a good vocation if it happens to discontinue its education before completing high school or higher secondary school. The future of the student in a technological age has to be given due consideration which constructing the mathematics curriculum.

Curriculum should be real and rationalistic: The curriculum should be real and should facilitate rational and original thinking.

Curriculum should emphasize learning to live rather than living to learn: Learning for the sake of learning is not to be encouraged. Curriculum should contain such information and experiences which can be assimilated and put into use for vital life situations.

Curriculum should help in preserving and transmitting our cultural traditions: Curriculum should contain such activities that help in preserving and spreading the culture of our nation.

Curriculum should be flexible and elastic: Curriculum should provide a variety of activities keeping in view the requirements of the students of different communities, religions,(rural or urban) and socio-economic strata.

Curriculum should emphasize attitudes rather than the acquisition of knowledge: The emphasis in curriculum activities should be more on the development of right attitudes among the students than on mere acquisition of knowledge.

The curriculum should be well integrated: The mathematics curriculum should conform to the curriculum of other subjects. There should be continuity and coherence within mathematics and with respect to other subjects as well. The learning of mathematics should help in viewing knowledge as a whole and not as segments of information.

The curriculum should provide both uniformity and variety: The curriculum should provide uniformity in terms of the content and objectives and variety in terms of the activities and experiences provided to the students based on their academic, social, intellectual and environmental requirements.

The curriculum should be useful to the students: While selecting topics due consideration should be given as to how useful the topics are in

i. Daily Life
ii. The study of other subjects
iii. Higher Education
iv. Selecting a good number of a vocations
v. Appreciating the role played by mathematics in the development of our culture and civilization.

B. Guidelines for Selecting the Topics in the Mathematics Curriculum

a) Cultural Perspective

Some ideas in mathematics that enable the student to appreciate and understand the culture and environment in which he is a part of, could find a place in the mathematics curriculum. In this category, the following topics may be included.

- Why and how a formula is developed
- Why and how an equation is solved
- The meaning of the graph
- The use of locus to solve position problems
- The use of indirect measurements with instruments
- How to unravel certain problems involving set-theoretic ideas

b) Participation in the technological, commercial and industrial civilization

Those topics which develop the mathematics skills and which are important for an individual to actively participate in his technological, commercial and industrial civilization should find a place in the mathematics curriculum.

c) Utility Value

Utility value is the most important criteria in selecting topics for the mathematics curriculum. J.W.A. Young has suggested the following criteria for the selection of the content of the syllabus. The selection should be such as:

- To exhibit most clearly and to the best advantage the mathematical type of thought.
- To help in a better understanding of the laws of nature.
- To bring out distinctly the mathematical relationships that exist in the social organization and in the activities of modern life.
- To give sufficient skill in the actual performance of mathematical processes to meet the future needs of the pupil.

- To permit the organization of the material into a homogenous whole, meeting the demands of scientific pedagogy.

Check Your Progress - 1
List the principles of Curriculum Construction

2.2.3.2. Organization of the Curriculum and Curricular choices at Secondary Stage

Exercise
Arrange the following topics in mathematics in the order you think that they should appear in the curriculum

(a) Fractions
(b) Progressions
(c) Operations of Mathematics (Addition, Subtraction, Multiplication and Division)
(d) LCM and HCF
(e) Sets

I am sure that you arranged the above topics in the following order, (c), (a), (d), (e) and (b). I am sure you used your discretion and decided that certain topics need to be learnt before the other topics to understand the following topics. Even the difficulty level of the topics also could have been your guiding force. As you answered the question one thing I am sure crossed your mind that is the organization of the topics. Similarly, the organization of the curriculum needs a detailed look into the different aspects related to the pupil and learning. Now let us discuss these aspects related to the organization of curriculum and also see what should be the curricular choices at the Secondary school level.

A. Organization of the Curriculum
After selecting the topics, the curriculum has to be organized, maintaining the mathematical sequence and continuity.

Principle of Curriculum Organization
The arrangement of topics in each class should be guided by the student's ability to grasp, assimilate, retain, and apply the mathematical concepts at a particular age level. The focus should be on the child and its capabilities rather than on the amount of information to be presented. The arrangement should be from concrete to abstract. The following principles should be kept in mind while organization the mathematics curriculum.

Principle of Correlation
While organizing the content in the mathematics curriculum the principles of correlation should be followed. The following four types of correlation should be considered.

i. Correlation with life
ii. Correlation with other subjects
iii. Correlation between different branches of mathematics
iv. Correlation between different topics in the same branch of the mathematics

Principles of Logical and Psychological Order
An integrated approach combining both logical and psychological order should be followed in the organization of the mathematics curriculum. The arrangement of the content should display sequential development of topics which is most appropriate for the student f the age-level.

Principle of Activity

Learning by doing makes learning more meaningful. The curriculum organization should take into consideration the type of activities that could be provided for the effective learning of the content. The activities that help in relating abstract mathematical concepts with concrete objects will induce enthusiasm and interest among the children. These activities could include:

Personal and home activities

- Vocational Activities
- Recreational Activities
- National Activities
- Community, civic and social activities.

Principle of Vertical Correlation

The content organized for a class should be based on the syllabus covered in the lower classes, and in turn, it should form the basis for the organization of the content in the higher classes. This is called vertical correlation. Topics arranged in any class also should follow the vertical correlation leading from simple topics to complex ones.

The Criterion of Difficulty: The organization of the content should be in the increasing order of difficulty. The difficulty level of a topic is to be judged from the pupil's point of view, based on the mental development and capabilities of the pupils.

Principle of Motivation*:* The organization of the content should enthuse the children to learn. The content presented should be challenging, interesting and exciting.

Adaptation to Individual Differences: The arrangement of the content for each class and level should cater to the needs of the different categories of children. There should be topics that are challenging for mathematically gifted students and topics suitable for the average and slow learner in mathematically gifted students and topics suitable for average and slow learners in mathematics. Similarly, the needs of students from rural and urban areas and different communities have to be given due weightage while arranging the mathematics curriculum.

B. Curricular Choices at Secondary Stage

In this stage that Mathematics comes to the student as an academic discipline. In a sense at, the elementary stage, mathematics education is (or ought to be) guided more by the logic of children's psychology of learning rather than the logic of mathematic. But at the secondary stage, the students begin to perceive the structure of mathematics. For this, the notions of augmentation and proof become central to curriculum now.

Mathematical terminology is highly stylized, self-conscious and rigorous. The student begins to feel comfortable and at ease with the characteristics of mathematical communication, carefully defined terms and concepts, the use of symbols to represent them, precisely stated propositions using only terms defined earlier, and proofs justifying propositions. The student appreciates how an edifice is built up, arguments constructed using propositions justified earlier, to prove a theorem, which in turn is used in proving more.

For a long, geometry and trigonometry have wisely been regarded as the arena wherein students can learn to appreciate this structure best. In the elementary stage, if students have learnt many shapes and know how to associate quantities and formulas with them, here they start reasoning about these shapes using the defined quantities and formulas.

Algebra, introduced earlier, is developed at some length at this stage. Facility with algebraic manipulation is essential, not only for applications of mathematics but also internally in mathematics. Proofs in geometry and trigonometry show the usefulness of algebraic machinery. It is important to ensure that students learn to geometrically visualize what they accomplish algebraically.

A substantial part of the secondary mathematics curriculum can be developed for consolidation. This can be and needs to be done in many ways. Firstly, the student needs to integrate the many techniques of mathematics she has learnt into a problem-solving ability. For instance, this implies a need for posing problems to students that involve more than one content area: algebra and trigonometry, geometry and mensuration and so on. Secondly, mathematics is used in the physical and social sciences, and making the connections explicit can inspire students immensely. Thirdly, mathematical modeling, data analysis and interpretation, taught at this stage, can consolidate a high level of literacy. For instance, consider an environment-related project, where the student has to set up a simple linear approximation and model a phenomenon, solve it, visualize the solution and deduce a property of the modeled system. The consolidated learning from such an activity builds a responsible citizen, who can later intuitively analyze the information available in the media and contribute to democratic decision making.

At the secondary stage, a special emphasis on experimentation and exploration may be worthwhile. Mathematics Laboratories are a recent phenomenon, which hopefully will expand considerably in the future. Periodic systematic will help in planning strategies for scaling up these attempts.

Check Your Progress - 2

List the aspects to be paid attention to while organizing a mathematics curriculum.

2.2.4. Let us Summarise

- **Principles of Curriculum Construction**
  - Principle of Child-Centeredness
  - Curriculum should provide a fullness of experience for the students
  - Curriculum should be dynamic and not static
  - Curriculum should be related to everyday life
  - It must take into account the economic aspect of the life of the people to whom an educational institution belongs
  - Curriculum should be real and rationalistic
  - Curriculum should emphasize learning to live rather than living to learn
  - Curriculum should help in preserving and transmitting our cultural traditions.
  - Curriculum should be flexible and elastic
  - Curriculum should emphasize attitudes rather than the acquisition of knowledge
  - The curriculum should be well integrated
  - The curriculum should provide both uniformity and variety

- **Guidelines for Selecting the Topics in the Mathematics Curriculum**
  - **Cultural Perspective:** Appreciation and understanding of the culture and environment which the student is part of
  - **Participation in the technological, commercial and industrial civilization:** develop mathematics skills which are important for active participation in technological, commercial and industrial civilization
  - **Utility Value:** Useful in day to day life

- **Organization of the Curriculum**

After selecting the topics, the curriculum has to be organized, maintaining the mathematical sequence and continuity

- Principle of Curriculum Organization
- Principle of Correlation
- Principles of Logical and Psychological Order
- Principle of Activity
- Principle of Vertical Correlation
- The Criterion of Difficulty
- Principle of Motivation
- Adaptation to Individual Differences

- **Curricular Choices at Secondary Stage**
  - The notions of augmentation and proof become central to curriculum
  - The student begins to feel comfortable and at ease with the characteristics of mathematical communication
  - Different branches of mathematics introduced during the earlier stages are developed at some length at this stage.
  - Emphasis on experimentation and exploration
  - Mathematics Laboratories

2.2.5. Answers to 'Check Your Progress' - 1 and 2'

Check Your Progress - 1
Principles of Curriculum Construction

- Principle of Child-Centeredness.
- Curriculum should provide a fullness of experience for the students.
- Curriculum should be dynamic and not static.
- Curriculum should be related to everyday life.
- It must take into account the economic aspect of the life of the people to whom an educational institution belongs.
- Curriculum should be real and rationalistic.
- Curriculum should emphasize learning to live rather than living to learn.
- Curriculum should help in preserving and transmitting our cultural traditions.
- Curriculum should be flexible and elastic.
- Curriculum should emphasize attitudes rather than the acquisition of knowledge.
- The curriculum should be well integrated.
- The curriculum should provide both uniformity and variety.

Check Your Progress - 2

After selecting the topics, the curriculum has to be organized, maintaining the mathematical sequence and continuity

- Principle of Curriculum Organization
- Principle of Correlation
- Principles of Logical and Psychological Order
- Principle of Activity
- Principle of Vertical Correlation
- The Criterion of Difficulty
- Principle of Motivation
- Adaptation to Individual Differences

2.2.6. Unit end Exercises

1. Explain the principles of curriculum construction supporting your answer with suitable examples.
2. State the guidelines for the selection of topics for the mathematics curriculum.
3. Explain the aspects to be considered when organizing a mathematics curriculum.
4. Explain the principles of curriculum organization.
5. Discuss the curricular choices to be made at the Secondary Stage for Mathematics.

2.2.7. References

1. Dr. Anice James (2019), Teaching of Mathematics. Neelkamal Publication Pvt. Ltd.
2. J.C. Agarwal, Essentials of Examination System, Vikas Publishing House Pvt. Ltd.
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Block 2 : School Mathematics Curriculum and Instruction

Unit 3 : Approaches to Curriculum Construction in Mathematics

Unit Structure

2.3.1. Learning Objectives

After completing this Unit, the student teachers will be able to

- Explain the approaches to curriculum organization;
- Explain the following Approaches;
  - ✓ Topical Approach,
  - ✓ Spiral Approach
  - ✓ Logical and psychological Approach
  - ✓ Unitary Approach
  - ✓ Integrated Approach
- Explain the characteristics of Modern Mathematics Curriculum; and
- Explain the Modern Approaches to Mathematics Curriculum.

2.3.2. Introduction

Approach is the way of looking at things. Curriculum construction and organization is a very important process in the hierarchy of teaching learning processes. It is the foundation for what will be delivered in class to the students. Hence it has to be sincerely and meticulously done, or else the whole system of teaching and learning can become flawed. Approaches form a very important guiding factor in the curriculum organization. What and how questions get a meaningful answer through the approaches. In the present unit, we shall discuss the various approaches to curriculum organization and also see the various characteristics of mathematics curriculum and modern approaches to the mathematics curriculum.

2.3.3. Learning Points and Learning Activities

2.3.3.1. Approaches to Mathematics Curriculum Organization-1

Exercise I

Take any mathematics topic of your choice and analyze it into sub-topics and decide in which class (8th, 9th or 10th standard) they should study the particular sub-topics your have identified and give reasons for the same.

Topic:

Subtopics	In which class should this subtopic be taught (8th, 9th or 10th)	Reason

As you answered the above question I am sure you had in mind different topics slotted for different classes keeping in mind the learning capacity of pupils. To arrange the topics more scientifically and psychologically the approaches of curriculum organization serve as guidance. Let us now go through the approaches of curriculum organization and also discuss the modern approaches to the mathematics curriculum.

A. Approaches to Curriculum Organization

There are different approaches to organize the mathematics curriculum. The important among them are:

1 Topical Approach
2 Spiral Approach
3 Logical and psychological Approach
4 Unitary Approach
5 Integrated Approach

1. Topical Approach: In the topical approach a topic once presented should be completely exhausted in the same class. This method demands that the entire topic, the portions easy as well as the difficult, should be covered in the same stage. This approach has many drawbacks as given below.

- This is not a psychological approach as the students are forced to learn many things for which they have no immediate need and relevance. For example, if 'Set Theory' is introduced in 8 standard, the entire unit on 'Set Theory' has to be completed in the same class without giving any consideration to the student's ability to learn.
- Some parts of the topic will be more complex and difficult for the learner to understand at the stage.
- It does not take into account the mental development of the students.
- It introduces a large amount of irrelevant matter, the use of which cannot be appreciated by the learner at the stage.
- Topics once completed receive no attention at later stages and there is every likelihood of them being forgotten.
- Topics are dealt with in watertight compartments and hence there is no chance of it being correlated with other topics or branches or subjects.
- Dealing with the same topic for a long time makes the learning dull, disinteresting, tedious, boring and tiresome.

It is not feasible to do any topic in its entirety in any class. Instead, a topic should be graded and arranged according to the increasing order of difficulty. Each part should be introduced at a stage when the student has the need to learn and the student has the intellectual development and capability to understand and appreciate what is presented to him.

2. Spiral Approach: The Cambridge Report (1963) on Mathematics Curriculum emphasized the importance of interrelating and interweaving the different mathematical topics to be taken up throughout the school period and envisaged the progressive broadening and deepening of the child's mathematical knowledge and insight by what is called the 'Spiral Approach'.

Therefore, contrary to the topical approach, the spiral approach demands the division of the topic into many smaller independent units to be dealt with, in order of difficulty, suiting the mental capabilities of children. It is based on the principle that a topic cannot be given an exhaustive treatment at one stage. To begin with, the elementary concepts are presented in one class, gaps are filled in the next class, and more gaps a year or two later, per the amount of knowledge which the students are capable of assimilating. For example, the unit on the 'Set Theory' can be split up into various subunits and arranged in the increasing order of difficulty. The elementary concepts like the definition of sets, and an introduction to set operations can be introduced in the 8^{th} standard; more complex properties on set operations, distributive laws, associates laws, D' Morgan's laws etc, can be presented in the 9^{th} standard and still more complex concepts in the 10^{th} standard and so on. Thus, in the spiral approach, the entire unit is gradually and successively introduced over the years.

The spiral approach has the following advantages over the topical approach,

- Subject matter is introduced in the increasing order of difficulty, per the needs and capacities of the students. It helps in a better understanding of the content.
- It satisfies the psychological needs of the students.
- The students can appreciate the relevance and significance of what they learn.
- It provides sufficient motivation for the students to learn.
- It provides opportunities for revision
- It provides opportunities to relate the topic with other topics, other branches and other subjects.

3. Logical and psychological Approach: In the development of a particular branch of mathematics, the mathematician is chiefly concerned with a logical rigorous treatment of the subject matter, whereas the teacher is usually concerned with its psychological organization and presentation. It is the curriculum organizer who is called upon to integrate the two approaches. It may be remarked that the two do not differ substantially.

There is no reason why the organization cannot be both logical and psychological. The happy combination of the two is very desirable and feasible. All thinking is psychological. Education should make it logical as well. Psychology throws light on the use of a topic for the student from the academic as well as practical point of view. It takes into consideration the power of understanding and grasping of pupils in a particular age level group. The order in which topics are to be taken will largely depend on its findings. Similarly, logic should be there. Psychology should decide what kind of logic is appropriate for the pupils of a certain age and which type of topics will be most suitable for the development of such logical thinking. Logic will help in maintaining the link and sequence of topics that are useful and meaningful for the child. In combination, these two viewpoints can make the subject matter interesting and comprehensible.

For Example, logically decimal fraction is to follow immediately after numerical notation, but psychologically it should not. This is said because the place value of the digits is the common principles involved. This principle of the place value of digits is involved also, multiplication, division, etc and any one of these topics can follow immediately after number notation. On the other hand, there is a principle not common to both. The idea of a fraction should be known before its notation, so both psychology and logic require that it should come after a fraction. There is no opposition between psychological and logical approaches.

4. Unitary Approach: The students learn mathematics with its different branches and topics in watertight compartments. So in the last few decades, the teachers in mathematics have searched for broad unifying principles that should be made the core of the mathematical course.

The most frequently mentioned unifying factor in arithmetic, algebra, geometry and indeed in all mathematical subjects is the function concept. If unifying principles are made the bases of organization, the courses will be greatly improved. Suppose that complete understanding of the function $y = ax + b$ be made one of the major division of first-year algebra, all the other concepts can be developed through this because the attainment of an understanding of this important function will depend upon many experiences. Such an organization enables the pupils to see the relationship between the various facts, processes and principles taught in the course. They know that each unit contributes to the course as a whole.

The question of time and size need not enter the organization of the unit. But for most high school pupils, it may take more than a month to assimilate. Experiences seem to show that a unit which can be studied in the four weeks is most suitable as to size. When it is not possible to finish it within that time the teacher should find a way of simplifying the unit either by transferring some of the materials to other units or by dividing it into two smaller units.

Characteristics of the unitary organization

- It organizes a body of facts, theorems or processes, closely related to one another and so organized as to contribute to the understanding of an important aspect of the course.
- It must be possible to present the theorems and process as a group in a form so definite that the learner may attain a conception of them before he undertakes the detailed study of the content of the unit.
- It must be possible to set up outcomes of the study so definite that they are clear not only to the teacher but also to the pupils.

I should be clear that there is no single type of organization that may be set up to determine an "only acceptable" list of units for a course. Units are organized to attain certain objectives of the teaching of mathematics. They must be subjected to experimentation and revision. A different set of objectives might influence considerably the choice of units.

Unitary organization makes teaching and learning purposeful and intelligent. Because the instructional materials are closely related to each other, they are easily retained. Economy of time and effort should be the result.

Steps in the Unitary Organization of the Curriculum

The following are the important steps in the unitary organization of the curriculum.

- **Setting up Objectives:** Setting up objectives to be attained is an important step in the plan of organizing a unit. The objectives should be measurable and observable. They should be clear and specific indicating the terminal learning outcome.

- **Preview of the Unit:** The preview of the unit is also another important step in organizing a unit. The purpose of the preview is to let the pupil view it as a whole before he begins to study details and to call his attention to the relations which exist among the various facts and principles that make up the unit. The methods of obtaining it may vary with different units and for different levels.

- **The study outline of the unit:** The purpose of an outline is to provide the pupil when he begins the study of the unit with directions as to what to study and how to do it most effectively. The teacher may include references to discussions or illustrations of principles studied in the unit. Mimeographed copies should be made available and each pupil should receive a copy.

5. Integrated Approach: The main aim of education is the acquisition of knowledge and the transfer of knowledge to study other subjects and to solve successfully the problems that arise in everyday life. Each subject in the curriculum aims at realizing these aims through different means. However, the knowledge and skills acquired through the subjects taught in a watertight compartment without relating to life and other subjects become redundant and meaningless. The study of every subject should highlight the unity of knowledge. While teaching any subject, the teacher can cite instances and examples to show that knowledge is a single integrated whole and the knowledge that one gain through courses like mathematics, physical/biological sciences, social sciences languages, arts and others constitute the whole. Such an integrated approach helps the students to get a holistic view of the entire school programme and thereby the study of each subject becomes more meaningful and significant.

Advantages of Integrated Approach

- Integrated Approach provides the students with ample opportunities to assess the number of situations where the knowledge that he has acquired can be applied. Thus the student realizes the importance of a particular subject, or topic and may become more interested in learning.
- Integration helps the students to view knowledge as an integrated whole and various disciplines as the constituent parts of the whole.
- Integration can lead t more effective learning as the same topic is dealt with in different situations and viewed from different angles.
- Integration helps to widen the mental horizon of the students. For instance, through an integrated approach, a student of mathematics becomes more familiar with other subjects, such as history, geography, art and architecture.
- Integration helps the students to appreciate the beauty and significance of other subjects and helps to develop a broad and comprehensive outlook.
- The student is able t transfer the learning from one situation to another with ease and the confidence as the situation requires.
- The integrated approach results in an economy of time and effort r subject and dealt with exhaustively.as the same topics that come under different subjects can be dealt in detailed manner.
- A teacher by following an integrated approach can adequately justify the unity of knowledge and help in a better understanding of the subject matter.

Check Your Progress - 1

1. “Topic once presented should be completely exhausted in the same class.” Which approach is this?
    - Topical Approach
    - Spiral Approach
    - Logical and Psychological Approach
    - Unitary Approach
    - Integrated Approach

2.3.3.2. Modern Approaches to Mathematics Curriculum Organization

Characteristics of Modern Mathematics Curriculum

Certain characteristics feature of a modern mathematics curriculum may be specified briefly as follows:

- Mathematics course materials should prepare the students for college but could be used with less talented students if they are given more time.
- New concepts and different points of view which are useful for the students should be there in the mathematics curriculum.
- Changes in the curriculum should help the students in meeting their present needs.
- The curriculum should provide an understanding of mathematics for future change and development.
- The curriculum should provide the application of mathematical structures and metric and non-metric relations in geometry.
- The curricular materials should involve experience with and appreciation of abstract concepts, the role of definitions, the development of precise vocabulary and thought and experimentation and proof.
- An emphasis on the structure of algebra for a clear understanding of the sound mathematics is essential in the mathematics curriculum.
- The mathematics curriculum should provide experiences to explore the behaviour of numbers and invent new numbers to describe new situations.
- The mathematics curriculum should be in harmony with the cultural experiences the children have at home and outside the school.
- The mathematics curriculum should be built on the mathematical experiences that the students already have gained.

Some Modern Approaches to Mathematics Curriculum
Brown et al (1997) described three different ways of viewing the mathematics curriculum

A. Cultural Induction
This way of viewing the curriculum is based on the work of Alan Bishop(1986) who experienced and studied in some detail the challenges of teaching and learning mathematics in a culture very different from his own. Bishop had suggested six cultural activities that drew heavily on, or are essential to, mathematics and which are crucial to each individual to be adequately inducted into the culture.

Counting: There are various types of counting:
- Using parts of the body as names
- Using counters and abstract names
- Using names alone

But counting is a composite act. An archetypal mathematical question is “How many” as in “How many different way?”. Many mathematical techniques amount to clever ways of counting things and many different definitions are about when two things are to be considered as different.

Locating: Finding your way around, locating yourself in space, orientation and special awareness, all mathematical ideas to enable precise calculation, communication and manipulation of ideas.

Measuring: Comparison is the essence of measuring. Developing from simple distinction through relative scales to absolute scales.

Designing: Artistic design is often mathematical. Example: pottery and fabric decoration, tiles pattern, stone carving. Tools intended for use also involve considerable design care.

Playing: Games have been part of human activity and most games have a mathematical basis that goes beyond the simple ideas following rules and working out consequences.

Explaining: Explaining and the related process of predicting are activities that lift human cognition above merely experiencing the environment and responding to it.

It would seem possible to view the mathematics curriculum through these heading within one culture or to use the heading within a multicultural environment as a device for enrichment.

A cultural approach to mathematics curriculum is of great significance to mathematics education in India. India being a country of vast cultural diversity, the mathematics curriculum should be rooted in the culture of the different sects of the Indian people. The mathematical knowledge that children learn in school should be based on the cultural experience that they have at home and outside the school. For effective and meaningful learning of mathematics, it is important to have a cultural basis fr the mathematics curriculum.

B. Mathematics Tools

This way of viewing the curriculum considers the entitlement of every child to achieve familiarity and facility in the use of the mathematical tools available in society. The focus of the teacher is on preparing tasks, which need to be supported by various tools. Initially, the pupil would have worked on problems with the aid of the tool and in the process learned how the tool works and what it can do. Eventually, while tackling a problem, the child reaches for the tool without being told to do so.

C. Essences

This way of viewing the curriculum considers that if pupils are to make sense of their mathematical lesson, then they need to be able to connect these experiences with what they already know. Mathematics lessons can truly start from where the pupils are by attending to root mathematical experiences or essences, evoking and building on them to engage with the desired mathematical content.

Check Your Progress - 2

1. List the characteristics of the Modern Mathematics Curriculum.
2. Name the different approaches to the modern mathematics Curriculum.

2.3.4. Let us Summarise

- Approaches to Curriculum Organization
    - i. **Topical Approach:** In topical approach a topic once presented should be completely exhausted in the same class.
    - ii. **Spiral Approach:** Spiral approach demands the division of the topic into many smaller independent units to be dealt with, in order of difficulty, suiting the mental capabilities of children.
    - iii. **Logical and psychological Approach:** Amalgamation of logical and psychological approach.
    - iv. **Unitary Approach:** Unification of different branches of Mathematics.
    - v. **Integrated Approach:** Integration of different aspects of teaching and learning.

- **Characteristics of Modern Mathematics Curriculum**
  - Prepare the students for college,
  - Meet their present needs.
  - Understanding of mathematics for future change and development.
  - Application of mathematical structures
  - Appreciation of abstract concepts, the role of definitions, the development of precise vocabulary and thought and experimentation and proof.
  - Provide experiences to explore the behavior of numbers and invent new numbers to describe new situations.
  - Harmony with the cultural experiences
  - Built on the mathematical experiences that the students already have gained.

- **Some Modern Approaches to Mathematics Curriculum**
  - **Cultural Induction:** Cultural activities that are essential to, mathematics and which are crucial to each individual to be adequately inducted into the culture.
  - **Mathematics Tools:** This way of viewing the curriculum considers the entitlement of every child to achieve familiarity and facility in the use of the mathematical tools available in society.
  - **Essences:** This way of viewing the curriculum considers that if pupils are to make sense of their mathematical lesson, then they need to be able to connect these experiences with what they already know.

2.3.5. Answers to 'Check Your Progress - 1 and 2'

Check Your Progress - 1
Topical Approach

Check Your Progress - 2

1. Characteristics of Modern Mathematics Curriculum
   - Prepare the students for college,
   - Meet their present needs,
   - Understanding of mathematics for future change and development,
   - Application of mathematical structures,
   - Appreciation of abstract concepts, the role of definitions, the development of precise vocabulary and thought and experimentation and proof,
   - Provide experiences to explore the behaviour of numbers and invent new numbers to describe new situations,
   - Harmony with the cultural experiences,
   - Built on the mathematical experiences that the students already have gained.

2. Some Modern Approaches to Mathematics Curriculum
   - **Cultural Induction:** Cultural activities that are essential to, mathematics and which are crucial to each individual to be adequately inducted into the culture.
   - **Mathematics Tools:** This way of viewing the curriculum considers the entitlement of every child to achieve familiarity and facility in the use of the mathematical tools available in society.
   - **Essences:** This way of viewing the curriculum considers that if pupils are to make sense of their mathematical lesson, then they need to be able to connect these experiences with what they already know.

2.3.6. Unit end Exercises

1. Explain the different approaches of Curriculum Organization.
2. What is a topical Approach? Explain.
3. Elaborate on the Logical and Psychological Approach.
4. Explain the integrated Approach with a suitable example.
5. List the characteristics of modern Mathematics Curriculum.
6. Explain the modern approach to Mathematics Curriculum.

2.3.7. References

1. Dr. Anice James (2019), Teaching of Mathematics. Neelkamal Publication Pvt. Ltd.
2. J.C. Agarwal, Essentials of Examination System, Vikas Publishing House Pvt. Ltd.
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4. Dr. S.K. Mangal, Pedagogy of Mathematics, Tandon Publications, Ludhiana.

Block 2 : School Mathematics Curriculum and Instruction

Unit 4 : Approaches in Teaching and Learning Mathematics

Unit Structure

2.4.1 Learning Objectives

After completing this Unit, the student teachers will be able to

- Explain the meaning of the term approaches to teaching mathematics;
- Explain the meaning of Inductive-Deductive Approach;
- Explain the meaning of Analytic-Synthetic Approach;
- Explain the meaning of Constructivist Approach; and
- Explain the teacher's role in adopting Constructivist Approach in teaching Mathematics.

2.4.2. Introduction

As we have seen that Approaches serve as a guiding force to the teaching-learning process, we must understand the different approaches that are useful for mathematics teaching. Approaches help the teachers select the right methods and techniques in teaching which can give the optimum learning experience to the pupils. Mathematics is a skill-based subject with ample opportunities to build your knowledge through construction and discovery. Hence the approaches chosen in teaching mathematics have to help the pupils to create and discover their knowledge with the help of teaching-learning techniques. In this unit, we shall discuss the approaches to teaching-learning which are useful in mathematics teaching.

2.4.3. Learning Points and Learning Activities

2.4.3.1. Approaches in Teaching Mathematics-1

Exercise I

Some procedures are listed below which are useful for teaching the topic "Quadrilaterals". Select one or a combination of them which you think is suitable to give your pupils optimum learning experience and also explain how you would use it effectively.

1. Show the pictures of Quadrilateral
2. Orally tell the students about Quadrilaterals
3. Give the students models of different quadrilaterals and ask them to identify their characteristics
4. Show a movie based on Quadrilaterals

5. Ask the students to identify shapes having four sides in the classroom
6. Ask the students to read the information on Quadrilaterals in the books provided.
7. Show animation on quadrilaterals

As you tried answering the above questions, I am sure you used a combination of procedures which you thought would give your students the optimum experience and I am also sure you might have added your ideas and improvised them to give a better experience to your students. There are several methods/strategies/techniques which we can use to teach mathematics effectively. When these techniques/strategies/methods are used in combination to fulfill a larger purpose they combine to form an approach of teaching. Now let us see some of the approaches that are useful in teaching Mathematics.

Meaning of the Term Approaches in Teaching Mathematics

The term approaches for teaching mathematics in its simple meaning stands for the type of approaches used by teachers or trainers in carrying out their teaching or instructional tasks. As far as teaching is concerned we may define it as a purposeful activity carried out by the teacher for guiding, directing and helping the learners in their pursuit of realizing the set teaching-learning objectives. In other words, what is planned and executed in the task of teaching is solely aimed at the learning of the learner. How one learns, can be explained based on the various theories of learning propounded by the psychologists belonging to different schools of psychology such as behaviorism, cognitivism, constructivism etc. The views expressed by the psychologists belonging to these schools about the learning of the children in one or the other learning situations may help the teachers in adopting one or the other approaches for carrying out their task of teaching in one or the other subjects of the school curriculum including mathematics.

Until now, the behaviouristic theories of learning have dominated the scene of classroom teaching-learning and teachers have been employing a behaviouristic approach for the teaching-learning of their students. In this approach knowledge and skill are directly poured into the minds of the students through lecturing or demonstration. Students remain most of the time passive listeners or silent spectators. They are, later on, required to reproduce or re-represent the reality of the facts communicated to them by the teacher. In this approach, learner is least bothered from where the communicated knowledge has come, he has to simply receive, revise, practice and then reproduce it in the form and shape communicated to him. Such teacher dominated and subject-centered approach for the teaching-learning of the school subjects including mathematics has been under better criticism like constructivist and discovery approaches based on the theories of learning and philosophy propagated by cognitivism and constructivism.

A. Inductive-Deductive Approach

Integration of more than one method comes to be an approach. Inductive-deductive approach is formed by the combination of Inductive and Deductive methods of teaching. Induction and Deduction go hand in hand.

a) Inductive Method

Inductive method is advocated by Pestalozzi and Francis Bacon. Inductive method is based on induction. Induction is the process of proving a universal truth or a theorem by showing that if it is true of any particular case, it is true of the next case in the same serial order and hence true for any such cases. So the technique is of making a transition from particular facts to generalizations about these facts is known as the process of induction. Thus, it is a method of arriving at a formula or a rule by observing a sufficient number of particular instances. If one rule applies to a particular case and is equally applicable to different similar cases, it is accepted as a general rule or formula. Therefore, in this method, we move from particular to general, from concrete instances to abstract rules and from simple examples to a complex formula. A formula or a generalization is arrived at through inductive reasoning.

b) Deductive Method

Deductive Method is based on deductive reasoning. Deductive reasoning is the process of drawing logical inferences from established facts or fundamental assumptions. In this method, the teacher presents the known facts or generalization and draws inferences regarding the unknown, following a network of reasoning. Therefore, in deductive method, one proceeds from general to particular instances and from abstract to concrete cases. This approach is not suitable for exploration, but appropriate for the final statement of mathematical results. In this method, we begin with a formula or rule or generalization and apply it to a particular case.

Induction and deduction are not opposite modes of thought. There can be no induction without and no deduction without induction. Inductive approach is a method for establishing rules and generalization and deriving formulae, whereas deductive approach is a method of applying the deduced results and for improving skill and efficiency of solving problems. Hence a combination of both inductive and deductive approaches is known as "inductive and deductive approach" is most effective for realizing desirable goals.

B. Analytic and Synthetic Approach

Analytic and Synthetic Approach is formed by the combination of Analytic and Synthetic Methods of teaching.

a) Analytic Method

The word 'analytic' is derived from the word 'analysis' which means breaking up or resolving a thing into its constituent elements. This method is based on analysis and therefore, in this method we break up the problem in hand into its constituent parts so that it ultimately gets connected with something obvious, or already known. Therefore, it is the process of unfolding of the problem or of conducting its operations to know its hidden aspects. In this process, we start with what is to be found out, (unknown) and then think of further steps and possibilities which may connect with the known and find out the desired result. Hence in this method, we proceed from unknown to known, from abstract to concrete and from complex to simple. In the analytic method the argument is that "To prove that B is true if A is true, it is sufficient to prove A is true."

b) Synthetic Method

'Synthetic' is derived from the word 'Synthesis'. Synthesis is the complement of analysis. To synthesize is to combine the constituent elements to produce something new. In this method, we start with something already known and connect it with the unknown part of the statement. Therefore, in this method, one proceeds from known to unknown. It is the process of combining known bits of information to reach the point where unknown information becomes obvious and true. In the synthetic method, the reasoning is as follows. "Since A is true, B is true"

Though both Analytic and Synthetic methods seem to oppose each other, they complement and support each other. As Arthur Schultze has pointed out "Analysis is the method of discovery; synthesis is the method of concise and elegant presentation". Therefore, it is preferable to use a combination of both methods, for the teaching and learning to effective, interesting and complete.

Check Your Progress - 1

1. What do you mean by Inductive-Deductive Approach
2. How is Analytic Approach different from Synthetic Approach?

2.4.3.2. Modern Approach for Teaching Mathematics

Constructivist Approach

Constructivism is first of all a theory of learning based on the idea that knowledge is constructed by the knower based on mental activity. Learners are considered to be active organisms seeking meaning. Constructivism is founded on the premise that, by reflecting on our experiences, we construct our understanding of the world consciously we live in. Each of us generates our own "rules" and "mental models" which we use to make sense of our experiences. Learning therefore is simply the process of adjusting our mental models to accommodate new experiences.

Constructivism is a set of assumptions about the nature of human learning that guide constructivist learning theories and teaching methods of education. Constructivism values developmentally appropriate teacher-supported learning that is initiated and directed by the students.

Constructivism is a learning theory focusing on the way people create meaning through a series of individual constructs or experiences. It emphasizes providing a learning environment where students can explore, test and acquire new knowledge on their own. Each person creates their mental models to deal with new information and experiences. Following are a few definitions of Constructivism.

Woolfok (2004:323) Constructivism represents "a view that emphasizes the active role of the learner in building, understanding and making sense of information."

Naylor and Keogh (1999): Constructivism represents an approach embedded with "the central principle that learner can only make sense of new situations in terms of their existing understanding."

Bruning, Schraw and Ronning (1999:215) The term constructivism more often emphasize " the learner's contribution to the meaning and learning through both individual and social activity,"

Lockhead (1985:4): What I see as critical is the recognition that knowledge is not an entity that can simply be transformed from those who have to those who don't. Knowledge is something that each learner must construct for and by himself. This view of knowledge as an individual construction is usually referred to as constructivism.

a) Characteristics of Constructivist Approach

Based on the definition cited above we may derive the following conclusions about the characteristics and unique features of the constructive approach associated with the philosophy of constructivism.

1. Constructivist approach focuses on knowledge getting or knowledge construction process rather than the product of knowledge directly communicated to the learners as happens in traditional classroom teaching based on the behaviorist approach.
2. It requires that the learners must take an active part in the knowledge getting process rather than being passive recipients of information or knowledge. The constructivism

main concern thus lies in shedding light on the learner as a key figure in the learning process rather than in wresting the power from the teacher.

3. As an active enquirer or discoverer of the knowledge, the students in the approach are required to make their meaning and constructing their knowledge while interacting within their classroom or social environment.
4. Constructivist approach argues and de3monstrates that a learner learns well when he is given opportunities to construct or discover the knowledge by himself.
5. Constructivist approach discourages the use of teacher-centered or subject-centered approach in the process of teaching-learning. Rather, it tries to encourage all the methods and techniques that are learner-centered and encourages understanding and particularly the reflective level of teaching-learning. It lays heavy emphasis on independent learning or solving the problems on the part of the learner so that he may be able to construct or discover his knowledge by his efforts on an individual or group basis. That is why, it is not unusual that constructivism is associated with the methods and strategies like discovery learning, inquiry approach, co-operative and constructivist learning and methods or strategies involving discussion, conservation, brainstorming etc.
6. In deriving their own meanings constructing their own knowledge at a time of their encounters in a particular teaching-learning situation, students may be generally found to make use of their prior knowledge and experiences as an essential tool for serving their purpose and since they differ in their stock of prior knowledge and experiences as well as in its processing, it is not unusual on their part to demonstrate wide variations in their output of meaning derivation or knowledge construction.
7. Constructivist approach although, puts much emphasis on construction or discovering the knowledge by the child himself by recommending the methods like discovery learning, inquiry approach, social investigation etc., yet it does not advocate to underestimate or deny the role of social interaction, cultural tools like language, assistance and guidance from the elders for enlightening the path of the learner in the accomplishment of his learning goals.

b) Teacher's Role in Adopting Constructivist Approach in Teaching of Mathematics

Following should be kept in mind by a teacher while adopting constructivist approach in mathematics teaching

1. The teacher should not take initiative or hurry in revealing the concept and processes of mathematics by himself but should inspire and provide opportunities to his students for constructing their knowledge or knowing about these things through their efforts.
2. Mathematics follows a logical sequence for the development of its concepts. The previously learned things help in constructing knowledge about the concepts following next in sequence. Therefore, while teaching a particular topic, the teacher should always try to become sure that the learner possesses all that is needed for the present learning i.e. constructing knowledge about the concepts related to the present topic. He should then persuade the learner to construct the required knowledge about the topic with the help of his previous knowledge, premises or essentials possessed by him on this topic.
3. The mathematical facts and the solutions of the problems should not be crammed but the facts should be known and problems should be solved by the students themselves by constructing the required knowledge through their efforts.
4. The essential thing in learning mathematics is not the acquisition of the knowledge but to acquire the way of constructing knowledge. The children therefore should be helped in getting equipped with necessary logic, reasoning, ways of thinking helpful in finding the solution of the mathematical problems and proof of the theorems as

well as building necessary concepts for understanding the structures and processes of mathematics.

5. When it is found that a student is feeling difficulty in constructing his knowledge based on his previous experiences and the reasoning and other procedural capabilities held by him, then the teacher hasto guide his path. However, here he should not hurry to present the solution of the problem or definition of the concept but to help him in providing the essentials or giving him some hint or clues for constructing his knowledge and achieving his objectives through his efforts.
6. Care should always be taken in arranging the learning experiences, and the related practice, drill, revision, project and assignment work in mathematics in such a way as to provide needed opportunities for the students to construct the knowledge by themselves, individually or in groups.
7. Adoption of constructivist approach on the part of teachers in mathematics demands to maintain necessary flexibility in arranging learning experiences for the learners in tune with their previous knowledge and capabilities of utilizing their intellect and mathematical reasoning for constructing the knowledge for themselves. Hence necessary efforts should be made to maintain the desired flexibility in planning learning experiences for the students on an individual or group basis while teaching mathematics with the adoption of constructivist approach.
8. Since prior knowledge and experiences on the part of learner are quite helpful for him to go ahead on the path of constructing and discovering the knowledge in his way, it is therefore quite imperative in adopting constructivism as a philosophy of imparting instruction on the part of mathematics teacher that the learning experiences provided to the children in mathematics should be customized to their prior knowledge and experiences. Besides that, the level of the learning experiences or teaching-learning encounters should be kept a little above so that the students may feel the necessity of the reconstruction of their prior knowledge and experiences independently or in the supervision or guidance of the teachers and elders.
9. Constructivist approach in its social constructivist format lays great emphasis on the roles of social interaction, encounters and cultural tools like language and social activities for helping the child in building his knowledge through his efforts. Consequently, a teacher of mathematics should explore all opportunities for gaining needed learning experiences by his students in the subject mathematics through
   a) Interaction and group discussion
   b) Project activities, drill and practice work in a small or large group
   c) Participation in social and community activities involving the use and application of mathematics.

As a whole thus while adopting constructivist approach for the teaching of mathematics due efforts should be made by the mathematics teachers.

i. To arrange learning experiences in such a way as to help the students to construct their own knowledge with their own efforts.
ii. To help the students in their efforts with the use of learner-centered methods like discovery method, inquiry training, concept attainment model, investigation method, a problem-solving method, project method, assignment method, discussion method, dialogue method (also called Socratian Method), co-operative and constructivism learning etc.

Check Your Progress - 2

Identify the unique features of Constructivist Approach. Plan a lesson for a topic of your choice using constructive approach.

2.4.4. Let us Summarise

- **Meaning of the Term Approaches for Teaching Mathematics:** The term approaches for teaching mathematics in its simple meaning stands for the type of approaches used by teachers or trainers in carrying out their teaching or instructional tasks. As far as teaching is concerned we may define it as a purposeful activity carried out by the teacher for guiding, directing and helping the learners in their pursuit of realizing the set teaching-learning objectives.

- **Inductive Deductive Approach:** Inductive-deductive approach is a combination of Inductive and Deductive method of teaching. Induction is the process of proving a universal truth or a theorem by showing that if it is true of any particular case, it is true of the next case in the same serial order and hence true for any such cases. Deductive reasoning is the process of drawing logical inferences from established facts or fundamental assumptions.

- **Analytic –Synthetic Approach:** Analytic and Synthetic Approach is formed by the combination of Analytic and Synthetic Methods of teaching. Analytic method is based on analysis and therefore, in this method we break up the problem in hand into its constituent parts so that it ultimately gets connected with something obvious, or already known. It is the process of combining known bits of information to reach the point where unknown information becomes obvious and true.
- **Constructivist Approach:** It assumes and pleads that the learners should play an active role in the process of learning for cognitive development by constructing their own knowledge on the basis of their past experiences and their present interaction with the environment.

While adopting constructivist approach for the teaching of mathematics due efforts should be made by the mathematics teachers.

1. To arrange learning experiences in such a way as to help the students to construct their own knowledge with their own efforts.
2. To help the students in their efforts with the use of learner-centered methods.

2.4.5 Answers to 'Check Your Progress - 1 and 2'

Check Your Progress - 1

1. Inductive-deductive approach is a combination of Inductive and Deductive method of teaching. Induction is the process of proving a universal truth or a theorem by showing that if it is true of any particular case, it is true of the next case in the same serial order and hence true for any such cases. Deductive reasoning is the process of drawing logical inferences from established facts or fundamental assumptions.
2. Analytic and Synthetic Approach is formed by the combination of Analytic and Synthetic Methods of teaching. Analytic method is based on analysis and therefore, in this method we break up the problem in hand into its constituent parts so that it ultimately gets connected with something obvious, or already known. It is the process of combining known bits of information to reach the point where unknown information becomes obvious and true.
3. Salient features of Constructivist Approach
    - Constructivist approach focuses on knowledge getting or knowledge construction process rather than the product of knowledge directly communicated to the

learners as happens in traditional classroom teaching based on the behaviorist approach.

- It requires that the learners must take an active part in the knowledge getting process rather than being passive recipients of information or knowledge.
- As an active enquirer or discoverer of the knowledge, the students in the approach are required to make their meaning and constructing their knowledge while interacting within their classroom or social environment.
- Constructivist approach argues and demonstrates that a learner learns well when he is given opportunities to construct or discover the knowledge by himself.
- Constructivist approach discourages the use of teacher-centered or subject-centered approach in the process of teaching-learning.
- Constructivist approach although, puts much emphasis on construction or discovering the knowledge by the child himself.

Check Your Progress - 2

Refer section 2.4.3.2

2.4.6. Unit-end Exercises

1. Explain the meaning of Approaches in teaching Mathematics.
2. Explain the Inductive –Deductive approach of teaching.
3. How is Inductive method different from deductive method? Explain,
4. Explain the Analytic-Synthetic Approach of teaching,
5. Explain the meaning of Constructivist Approach quoting suitable definitions.
6. Explain the characteristics,
7. Explain the teacher's role in adopting Constructivist Approach in teaching Mathematics.

2.4.7. References

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Block 2 : School Mathematics Curriculum and Instruction

Unit 5 : Strategies of Teaching and Learning Mathematics

Unit Structure

2.5.1. Learning Objectives

After completing this Unit, the student teachers will be able to

- Explain the meaning of Techniques, Strategies and Methods of Teaching Mathematics;
- Explain various strategies of teaching Mathematics, namely; and
    - Drill
    - Homework
    - Correction of Homework
    - Oral Work
    - Written Work
    - Group Work
    - Self Study
    - Supervised Study
    - Review

2.5.2. Introduction

Techniques and Strategies form the building blocks of teaching and learning. To introduce and teach the concepts to pupils and thereafter for meaningful assimilation of those concepts by the pupils, a teacher will have to use many effective strategies and help him/her reach the goal. These strategies have to be realistic and simple so that the learning is smoothly feasible. Several strategies are useful in teaching mathematics to pupils. In this unit, we shall first see what is the meaning and difference between techniques, strategies and methods and then understand various strategies that are useful in mathematics teaching and learning.

2.5.3. Learning Points and Learning Activities

2.5.3.1. Strategies in Teaching and Learning Mathematics-1

Exercise I

"A cat has fallen into a well" What strategies would you use to bring the cat out of the well.

You would surely have thought of many ways to take the cat out. You could send a basket down, or you get into the well or call a professional rescuer etc. Similarly, several strategies could be used in teaching mathematics content. Let us discuss them.

Meaning of Techniques, Strategies and Methods of Teaching Mathematics

In Mathematics teaching, teaching techniques are such aids that are used to make the lesson interesting, to explain the content and to remember it by heart during the teaching-learning process. Techniques are not directly linked with the teaching objectives, but they are linked with teaching methods, while methods are directly linked with teaching objectives. On the other hand, teaching strategies are purposefully conceived and determined plan of action. Thus teaching or instructional strategies refer to a pattern of teaching acts that serve to attain certain outcomes and to guard against others. According to Smith "The term strategy refers to patterns of acts that serve to attain certain outcomes and guard against certain others." Strategy requires some sort of planning. When faced with a new situation, you would probably use a strategy. It is a plan of action designed to achieve an overall aim.

Method is a wider term. It includes techniques and strategies of teaching. Different strategies may be adopted in the following method. Teaching strategies may include different techniques for teaching. Various techniques may be used within the same strategy and method. A teaching strategy assumes that teaching is science while a method assumes that teaching is an art. The term teaching strategy owes its origin to military science whereas method is a term of Pedagogy. Hence Teaching strategies and techniques are used to make the teaching effective successful and interesting.

Various Strategies and Techniques of Teaching Mathematics

A. Drill in Mathematics: Drill work is based on psychological principles such as learning by doing and the law of exercise. Drill plays a prominent role in learning because it affords a convenient and fairly efficient medium for the rapid memorization of details and the automation of processes. Drill must be recognized as an essential means of attaining some of the desired controls, just as a strong emphasis upon concepts and meanings must be regarded as essential for understanding. Both are necessary and neither alone is sufficient.

Drill provides an opportunity for self-learning and improvement. The speed and accuracy in mathematics cannot be possible without drill work. Be certain that understanding precedes the drill. Otherwise, the practice becomes an exercise in academic futility and no one benefits.

Points to be followed in making Drill Effective in Mathematics Teaching

- Drill exercises should be conducted in such a manner that pupils can work at different rates and different levels according to their abilities.
- Drill exercises should be brief and distributed over a period of time.
- Drill should consist of several distinct activities involving different strategies of learning.
- A variety of problems will make the drill interesting.
- The drill work should be progressively more challenging.
- Drill should follow developmental and discovery stages of learning and be used to reinforce and extend basic learning.
- Drill exercises should contain enough material to keep all the students profitably occupied throughout the drill period.

- Sufficiently diversified material to provide worthwhile and simulating practice for students of different attainments and capacities.
- To be most effective, drill exercises must be specific.
- Drill should be concentrated upon particular skills or even on particular details of the operation.
- Efforts should be made to detect mistakes in children's work and eliminate them at the outset.
- It is of extreme importance to supervise closely the initial work of the students on any new process.
- After the children have done some practice, the teacher should try to elicit a summary of what has been learned in the classroom.
- The summary can also be developed by asking questions reacted to the concept developed by asking questions related to the concept developed in the classroom.
- The children could be instructed to carry out some of the operations in each of the preceding problems as evidence of the level of achievement that has been attained by the class.

B. Homework in Mathematics

The homework in mathematics may consist of some problems based on facts taught in the classroom. The student may be asked to learn certain principles definitions, facts, draw graphs, charts, tables etc. By giving homework means creating in the children a study environment at home. The nature and amount of homework should be given according to the capacities of the children. Home should be assessed as a part of internal assessment and proper weightage should be given.

Moreover, make sure each exercise serves a definite purpose. Do not assign only problems on the newly developed topic of the day. Such type of work could result in the children trying to work on a skill or concept that they do not thoroughly understand. They may even develop some misconceptions, and too much practice in the assigned subjects could result in their fixing these misconceptions in their mind. It should be remembered to keep homework brief. Most of the children are given some homework in every class they have; if each teacher gives one hour of homework, this can result in many hours of work at home.

Purpose of giving Homework to Pupils

- It utilizes the leisure time of the child.
- It cultivates the habit of regularity and hard work among the children.
- It provides the opportunity for independent work.
- It provides an opportunity for the application and practice of the gained knowledge.
- It supplements classroom teaching.
- It acts as a link between parents and teachers.
- It creates an environment of school feeling at home among the children.

C. Correction of Homework

The correction of homework in mathematics is very important. If it remains unchecked, it does not fulfill its purpose. However, regular correction of homework in mathematics is also a very difficult task for a teacher. To do some justice to this task, the teacher may have sample checking every day. The teacher may indicate the correct answer and solution on the blackboard. The teacher may also introduce surprise checking and cross-checking or mutual checking by the exchange of notebooks.

Points to be kept in mind while correcting Homework

- Stress should be given to neat and clean work.
- Transcription should be reduced to a minimum.
- Gradation is very necessary.
- If possible remarks may be written and suggestions should be given to the student.
- The teacher should indicate the mistakes by suitable remarks in ink or pencil of a different colour.
- The corrections made by the teacher ought to be written by the children. This may better be checked by the teacher.
- The teacher may make a list of common errors and discuss them in the class.
- The procedure should be to move step by step. No step should be omitted.

D. Oral Work in Mathematics

Oral work results in saving time and effort through the omission of certain steps. Oral work helps us in mental calculations. It gives a quick and easy start to the process of learning. The lesson can be introduced through short, easy and appropriate oral questions. Oral question makes the lesson easily comprehensible and makes the process of learning very clear. Oral questions can be graded according to t difficulty in a better manner. This demands the power of careful listening, visualization, quick thinking and decision making. It is very easy to discover the weakness of the child through oral work in mathematics and his mistakes can be rectified. Hence all new processes and methods should be introduced initially in the pupil's mind orally. Thus they will develop an interest in the new material. When oral work has been done, written work may follow it, because it is an admitted fact that, "Reading make a full man, conference a ready man and writing an exact man." Therefore oral work must be supplemented by written work.

E. Written Work in Mathematics

To attain precision and accuracy, written work is essential in mathematics. Simple oral discussions are not enough. Moreover, based on the psychology of the visual and auditory types it is evident that all children cannot benefit from oral work alone. Therefore, oral work must be supplemented by written work. It is better to join both of these techniques together, one without the other is vague and purposeless. They both combine to make the process of instruction complete and an attempt to associate them would be ludicrous. Hence, mental work has to be combined with written work and both oral and written work must be included in the teaching and learning.

Written work enables the teacher to know the amount of work done by his pupils. It helps in testing the knowledge imparted orally. Moreover, in written work we can, make the students work in accordance with proper rules, processes and principles.

Importance of Written work

- Learning by this technique is retainable for a longer time.
- Memory of the child can be tested.
- Expression and writing power to the student can be checked properly.
- In case needed, suggestions to improve the handwriting can be given to the students.
- Confidence in the process of learning can be developed.
- Mistakes can be checked properly.
- Practice of the learnt material can be carried out easily.
- Verbalism can be reduced in the process of learning and teaching.
- Speed of writing can be improved.
- Record of the learnt material can be kept for reference.

F. Group Work in Mathematics

It was widely used at Nalanda University. The Greek scholars used to discuss various problems and issues with their disciples. In mathematics, there is ample scope for group work. In case the teacher teachers by activities, projects, assignments or practical work, the pupils find many opportunities for group work. In earlier strategies, oral and drills were conducted generally in groups. The principles vowels underlie in group work are:

- Principle of active participation.
- Principle of freedom for work.
- Principle of equal opportunities.

The need for Group Work in Mathematics

- To consider, examine and investigate the various aspects of a question, topic or problem.
- Group Work for doing homework.
- Thoughtful consideration of the relationships in topic or problem under group study.
- The relationships are analyzed, compared and evaluated and conclusions may be drawn.
- Table recitation
- Collection of mathematical data from the field.
- Preparation of mathematical models or some mathematical material.
- To exchange the ideas, opinions and experiences of the children.

Characteristics of Group Work

- The group should be homogeneous in the matter of intelligence and level of achievement.
- Two heads are better than one. Exchange of ideas and opinions.
- It makes competitive c-operation among group members.
- Utilization of experience and learning from oneanother.
- The nest age in which group may be most profitable is the age from eight to twelve years.
- The purpose of group work must be clear.

Check Your Progress -1

Select a topic where you can follow group work and explain the procedure you would follow while teaching it.

2.5.3.1. Strategies in Teaching and Learning Mathematics - 2

G. Self-Study in Mathematics

Self-Study means an individual's independent study. The individual learns and studies by himself without any help from outside. So it is a habit of independent study by which the children can solve the mathematical problems with their effort. Self-study can be made more effective and systematic by giving regular homework or assignments. Preparation for projects, debates, discussions, seminars and other competitions also requires self-study. It is essential to ensure regular progress in mathematics. In this technique, the children learn to make use of their knowledge and experiences in tackling various problems. It also develops self-confidence and self-independence in the children so that they do not hesitate in tackling problems.

Importance of Self Study

- It develops a sense of responsibility and regularity.
- The child works/studies independently.
- The child gets the opportunity to use his knowledge and experiences.
- Self-study discourages the habit of cramming.
- It helps in proper utilization of leisure time.
- It develops a heuristic and problem-solving attitude in the children.
- It is the best way to supplement class work.
- It develops the habit of practice/drill
- Self-study develops initiative and independent thinking in the children.
- The child is his guide and supervisor.
- It is required for the preparation of debates, discussions, seminars, examinations and other competitions.
- It is essential for the regular progress of the child.
- It is also essential to complete the other assignments in mathematics.

H. Supervised Study in Mathematics

It is an important technique for teaching mathematics. Marrison has presented it for the teaching of the understanding level. This technique is based on the principle of activity and individual differences. Dr. N.R. Swarup Saxena and Dr. S. C. Oberoi have remarked that in this technique, keeping in view the individual differences, every child is provided with an opportunity to d his respective task and study. The teacher solves his problems by supervising his task as a friend, helper and guide. Therefore, in this technique both the teacher and child remain active.

Supervised study introduces the regularity in work and ensures sustained progress. The mistakes and difficulties of the children can be removed on the spot. This technique develops the habit of self-study in students. In a supervised study, every child has to devote his prescribed time compulsorily for self-study. This technique removes the demerits of the traditional techniques of teaching mathematics such as illustrations, explanation etc. It creates a formal atmosphere for the self-study. This is a well-known feature of public schools where the resident children are to assemble at one place at night to study under the guidance of their teachers. In this technique, the child learns according to his abilities and capacities. The technique may be useful when all the pupils of the class are to be observed and when backward students are to be observed.

Characteristics of Supervised Study

- Every child gets an opportunity to learn according to his abilities and capacities.
- This technique develops the habit of self-study.
- It is based on the principle of activity and individual differences.
- The teacher’s presence makes the atmosphere more disciplined.
- The teacher serves as a guide.
- Both the teacher and child remain active throughout the study.
- It develops a group feeling in the mind of the child.
- There is no need of giving additional homework to the child.
- It helps to develop many qualities like self-reliance, hardwork and self-confidence.
- It develops the habit of regularity in work and ensures sustained progress.

I. Review in Mathematics

Sometimes a review is identified with drill work because they are both characterized by repetition and both aim at the fixation of concepts, relationships or reactions. Review aims not only at the fixation and retention of details but also at the thoughtful organization of important things in a unit or a chapter so that the relationship of the various parts to each other and the whole unit may be understood clearly, while drill is mainly aimed at the automatization of relatively detailed processes and reactions.

The function of the review is indeed to make recall more certain and more effective. Review means "re-view" or a new look, at the unit which has been studied, rather than through reducing reactions to the plane of automatic response. Review emphasizes thought and meaning rather than habit formation. Therefore review and drill have some things in common also have some differences. Both the review and drill are very important in the study of mathematics.

The children need to be taught how to review material just as they need to be taught how to study. They cannot review effectively without definite instructions. The task of helping children to plan their review work is the responsibility of the teacher. Butter and Wren, in their book entitled the teaching of Secondary Mathematics (IV Edition) has written that,

"Review work may be incidental in the sense that it may be integrated with the other work of the course, or it may be specialized by making it the primary feature and objective of a particular assignment. Both of these types of reviews are necessary for the most effective teaching and learning. Incidental type of review is especially valuable for the gradual building up and clarification of concepts through repeated reference and through continual reapplication in those situations in which they play component parts. Though making the necessary associations of the ideas in the unit, he will be aided not only in remembering them but in understanding them and appreciating theirs under the relation. Hence, To minimize forgetting of the acquired knowledge, a systematic or periodic review is very necessary. Review can take the form of checking verification or confirmation of the knowledge acquired by the children.

J. Assignments in Mathematics

Assignment is the work given to the students either before the lesson or after the lesson or and it may be completed at school or home. Assignment is a sort of undertaking or commitment on the part of the learner. The child undertakes upon himself the responsibility of carrying out the work assigned to him. Assignment should be brief so that pupils will be more willing to try to do it. Assignment in mathematics includes two different types of problems – Repetitive problems and Review problems.

Repetitive problems are based on new work. By assigning problems on several different topics, the teacher provides the child with a variety in his assignment, which might add some interest to the task of assignment. The repetitive problems serve to emphasize some aspects of what has been newly learned in the classroom that day. Thus these problems provide the child an opportunity to see if he has mastery over what was taught.

The review problems are that which spiral back over the skills and concepts learned in previous topics hence they are also called a spiral assignment. The spiral assignment contains both repetitive and review problems. These are the problems that do not allow the child to forget the mathematics he has learned previously. So these problems should be selected very carefully.

These problems may have some bearing on the work done that day in the class, or they may simply be review problems whose solution is intended to bring back the child's facility in working with previously learned material. Both sorts of review problems help prevent the forgetting phenomenon. In the daily assignments one or two verbal or word problem should be included and assignment should not be used as a punitive device.

The problems involving the concept should be included in the assignment. When the assignment is presented to the class at the end of the period, the teacher should know which problems have been touched upon in the classroom. It must be remembered that, if an assignment is worth giving then it should be worth checking. This enables the teacher to see which child is having problems with the work and at the same time, which problems are causing difficulties for the entire class.

Purpose of Assignment in Mathematics

- To solve mathematics problems.
- To prepare illustrations for a topic.
- To collect mathematical data.
- To understand a proposition or a group of propositions.
- To trace out the background of a mathematical problem/concept.
- To formulate problem on a topic/concept.
- To carry out some mathematical projects.
- To apply mathematical knowledge in solving the problems.
- To create interest in mathematics to develop the skills of problem-solving.
- To develop the habit of practice/exercise.
- To correlate the experiences and previous knowledge of the child.

K. Brainstorming in Mathematics.

Brainstorming is a democratic and problem-centered technique. It is based on the modern theory of generalization of task. In this technique, the content is largely determined by the children. Brainstorming created situations for students and teacher interaction and both remain active in teaching. This technique encourages creativity among children. Brainstorming is based upon the assumption that a child can learn better in a group rather than in individual study. By this, the higher order of cognitive and affective objectives can be achieved.

In this technique, the teacher assigns a problem to all the children. All the children think over the problem independently and then they discuss and arrange a debate. The children are asked to express their views and ideas which come to their mind frankly. It is not necessary that whether the ideas and views of the children are meaningful or not. The teacher writes children's views and ideas on Black-board. In this way, the problem is solved through Brainstorming.

Hence Brainstorming is based on the principle that the children can be provided with more and more knowledge through interaction. Therefore, such means are used which create movement in the minds f the children of the class for mutual consultation, logic, reasoning and discussion To solve some mathematical problems.

So the teacher should use brain-storming in the classroom as much as possible so that the qualities like self-confidence, originality, creativity, reasoning etc. may develop in the mind of the child.

Importance of Brain Storming

- It is a problem-oriented strategy of teaching-learning.
- It helps to achieve a higher order of cognitive and affective objectives.
- It is the democratic technique of teaching.
- It provides more ideas and views of the child.
- It is more creative and encourages the originality of ideas.
- It creates the situation for more independent study, thinking and reasoning.
- It increases the knowledge of the child.
- It makes classroom interaction more effective.
- It has both psychological and educational importance.
- The child selects ideas most likely to lead to the solution so that the habit of decision making is developed in the mind of the child.
- It develops problem-solving ability.

Check Your Progress - 2

Explain the different strategies that can be used in teaching Mathematics

1.5.4. Let us Summarise

- **Meaning of Techniques, Strategies and Methods of Teaching Mathematics**
  - Techniques are not directly linked with the teaching objectives, but they are linked with teaching methods, while methods are directly linked with teaching objectives.
  - Teaching strategies are purposefully conceived and determined plan of action. According to Smith "The term strategy refers to patterns of acts that serve to attain certain outcomes and guard against certain others." Strategy requires some sort of planning.
  - Method is a wider term. It includes techniques and strategies of teaching.

- **Various Strategies and Techniques of Teaching Mathematics**
  - **Drill in Mathematics:** Drill work is based on psychological principles such as learning by doing and the law of exercise. Drill plays a prominent role in learning because it affords a convenient and fairly efficient medium for the rapid memorization of details and the automation of processes.
  - **Homework in Mathematics:** The homework in mathematics may consist of some problems based on facts taught in the classroom. The student may be asked to learn certain principles definitions, facts, draw graphs, charts, tables etc. By giving homework means creating in the children a study environment at home.
  - **Correction of Homework:** Correction of homework is an essential part of mathematics learning. The work done by students is corrected for mistakes and given feedback immediately.
  - **Oral Work in Mathematics:** Oral Work is a process where a pupil's learning is expressed in the oral form. Oral work results in the saving of time and efforts through the omission of certain steps. Oral work helps us in mental calculations. It gives a quick and easy start to the process of learning.
  - **Written Work in Mathematics:** To attain precision and accuracy, written work is essential in mathematics.
  - **Group Work in Mathematics:** In group work pupils work in groups to complete the task assigned by the teacher.
  - **Self-Study in Mathematics:** Self Study means an individual's independent study. The individual learns and studies by himself without any help from outside. So it is a habit of independent study by which the children can solve the mathematical problems with their effort.

- **Supervised Study in Mathematics:** This technique is based on the principle of activity and individual differences. In this technique, keeping in view the individual differences, every child is provided with an opportunity to d his respective task and study. The teacher solves his problems by supervising his task as a friend, helper and guide.
- **Review in Mathematics:** Review aims at the fixation and retention of details but also at the thoughtful organization of important things in a unit or a chapter so that the relationship of the various parts to each other and the whole unit may be understood clearly.
- **Assignments in Mathematics:** Assignment is the work given to the students either before the lesson or after the lesson or and it may be completed at school or home.
- **Brainstorming in Mathematics:** Brainstorming is a group creativity technique by which efforts and made to find a conclusion for a specific problem by gathering a list of ideas spontaneously contributed by its members. In this technique, the teacher assigns a problem to all the children. All the children think over the problem independently and then they discuss and arrange a debate.

2.5.5. Answers to 'Check Your Progress - 1 and 2'

Check Your Progress - 1
Refer Section 2.5.3.1.

Check Your Progress - 2
Various Strategies and Techniques of Teaching Mathematics

- Drill in Mathematics
- Homework in Mathematics
- Correction of Homework
- Oral Work in Mathematics
- Written Work in Mathematics:
- Group Work in Mathematics
- Self-Study in Mathematics
- Supervised Study in Mathematics
- Review in Mathematics
- Assignments in Mathematics
- Brainstorming in Mathematics

1.5.6. Unit end Exercises

1. Explain the meaning of techniques, strategies and methods and derive the difference between them.
2. Explain various strategies of teaching mathematics.
3. Explain Drill Work as a strategy in teaching mathematics.
4. How can homework be a strategy in teaching Mathematics? Explain.
5. Explain 'Correction of Homework' teaching strategy.
6. Elucidate Oral Work.
7. Explain how you will use written work in teaching Mathematics.
8. How effective can Group Work be in teaching Mathematics? Explain
9. Explain Self Study as a teaching strategy.
10. Write a short note on 'Supervised Study'.
11. With an example explain how 'Review' can be effective in teaching Mathematics.

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Block 2 : School Mathematics Curriculum and Instruction

Unit 6 : Difference between Teaching of Mathematics and Teaching of Science

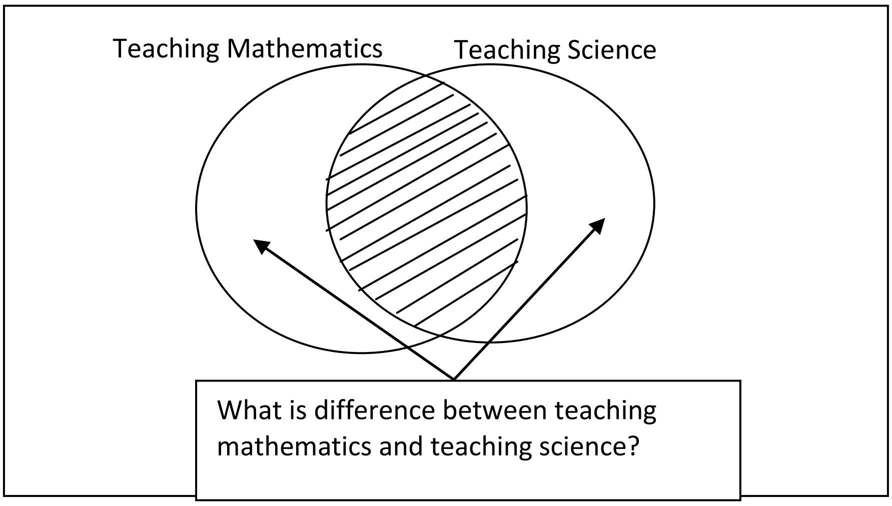
Unit Structure

2.6.1. Learning Objectives

After completing this Unit, the student teachers will be able to

- How mathematics, science and technology are related;
- Nature of mathematics and its implications for mathematics Pedagogy;
- Nature of science and its implications for science Pedagogy; and
- Difference between teaching mathematics and teaching science.

2.6.2. Introduction



There are lots of commonin teaching mathematics and teaching science. Most of them believe that teaching mathematics is the same as teaching science. There are a few important differences are there between teaching mathematics and science. To understand the pedagogical aspects of any discipline it is imperative to understand the philosophy of that discipline. In other words, one needs to know the nature of discipline to understand how to teach that particular discipline. This is because the nature of discipline is the major and only clue to understand the pedagogical implications of that discipline. Hence a teacher teaching a particular discipline to the students should inevitably understand the nature of the discipline he or she is teaching.

Contemporary society lumps mathematics and science as one thing, but they are not the same. Mathematics is based on abstractions and relationships between abstractions. Abstractions in mathematics are generally absolute truths, meaning the abstraction may be true. Very few things that are accepted in mathematics get retracted later. New abstractions can be formed from existing ones, usually from those that are absolute truths, and these new abstractions can be formed by simply sitting at a desk and thinking about it.

The basis of science is that all knowledge is based on experience derived from the senses. One observes something about the natural world and tries to create his model. Verification of a model is usually not absolute, and through repetition and logic, something is "believed" to be true when as far as anyone can tell there's no evidence that it is false. Science is often not based on absolute truths; many things in science that are once accepted get retracted from days to centuries layer.

Therefore, to understand the difference between the teaching of science and mathematics we need to analyse the nature of these subjects. The teaching of these subjects has some aspects in common and some aspects significantly differ from each other.

2.6.3. Learning Points and Learning Activities

2.6.3.1. Teaching of Mathematics and Teaching of Science

Let us analyse the nature of these subjects and simultaneously deduct their implications for teaching them.

Science and mathematics are intimately connected, particularly in fields such as chemistry, astronomy and physics. Students who can't master basic arithmetic skills will struggle to read scientific charts and graphs. Mathematics is also important in practical sciences, such as engineering and computer science.

Because of its abstractness, mathematics is universal in the sense that other fields of human thought are not. It finds useful applications in business, industry, music, historical scholarship, politics, sports, medicine, agriculture, engineering, and the social and natural sciences. The relationship between mathematics and the other fields of basic and applied science is especially strong. This is so for several reasons, including the following:

- The alliance between science and mathematics has a long history, dating back many centuries. Science provides mathematics with interesting problems to investigate, and mathematics provides science with powerful tools to use in analyzing data. Often, abstract patterns that have been studied for their own sake by mathematicians have turned out much later to be very useful in science. Science and mathematics are both trying to discover general patterns and relationships, and in this sense, they are part of the same endeavor.
- Mathematics is the chief language of science. The symbolic language of mathematics has turned out to be extremely valuable for expressing scientific ideas unambiguously. The statement that $a=F/m$ is not simply a shorthand way of saying that the acceleration of an object depends on the force applied to it and its mass; rather, it is a precise statement of the quantitative relationship among those variables. More

important, mathematics provides the grammar of science - the rules for analyzing scientific ideas and data rigorously.

- Mathematics and science have many features in common. These include a belief in understandable order; an interplay of imagination and rigorous logic; ideals of honesty and openness; the critical importance of peer criticism; the value placed on being the first to make a key discovery; being international in scope; and even, with the development of powerful electronic computers, being able to use technology to open up new fields of investigation.
- Mathematics and technology have also developed a fruitful relationship with each other. The mathematics of connections and logical chains, for example, has contributed greatly to the design of computer hardware and programming techniques. Mathematics also contributes more generally to engineering, as in describing complex systems whose behavior can then be simulated by computer. In those simulations, design features and operating conditions can be varied as a means of finding optimum designs. For its part, computer technology has opened up whole new areas in mathematics, even in the very nature of proof, and it also continues to help solve previously daunting problems.

Check Your Progress - 1
Explain the relationship between mathematics and science

2.6.3.2. Nature of Mathematics and its implications for mathematics Pedagogy

Mathematics is the gateway of all science. Mathematics reveals hidden patterns that help us understand the world around us. Mathematics is the science of patterns and relationships. As a theoretical discipline, mathematics explores the possible relationships among abstractions without concern for whether those abstractions have counterparts in the real world. The abstractions can be anything from strings of numbers to geometric figures to sets of equations. The nature of mathematics are enlisted in the following points:

- Mathematics is an exact science. Mathematical knowledge is always clear, logical systematic and may be understood easily.
- It is the science of space, numbers, magnitude and measurement.
- Mathematics involves conversion of abstract concepts into concrete form.
- It is the science of logical reasoning.
- It helps the man to give an exact interpretation of his ideas and conclusion.
- Mathematics is that science which is a by-product of our empirical knowledge.
- Mathematical propositions are based on postulates and axioms from our observations.
- It may exhibit an abstract phenomenon in the concrete. Thus abstract concepts may be explained and understood with the help of mathematics.
- It is related to each aspect of human life.
- Mathematical knowledge is developed by our sense organs therefore it is exact and reliable.
- The knowledge of Mathematics remains the same in the whole universe, everywhere and every time. It is not changeable.
- The knowledge of mathematics has no doubt. It provides a clear and exact response like yes or no, right or wrong.
- It involves inductive and deductive reasoning and can generalize any proposition universally.
- It helps the self-evaluation.

The special role of mathematics in education is a consequence of its universal applicability. The results of mathematics; theorems and theories; are both significant and useful; the best results are also elegant and deep. Through its theorems, mathematics offers science both a foundation of truth and a standard of certainty.

Mathematics as a language needs to be used to communicate ideas. Communication must occur in school and society between and among individuals pertaining to quantitative topics. Precision is involved. Mathematics is perhaps the most objective of all academic disciplines whereby individuals agree, as a whole, with the quantity being discussed. It attempts to be precise with the quantity being considered. Among others, mathematical ideas may be expressed orally and in diagrams, charts, tables, formulas, library books, graphs, and written work.

Students need to make connections in terms of the use of mathematical subject matter. They need to perceive how mathematics relates to the self, as well as others. Problem-solving is a vital skill for all to develop. Developmentally and at increasing levels of difficulty, students must be able to solve personal mathematics problems. Logical thinking is necessary here as well as dealing factually with quantitative information. Critical thinking is inherent when separating facts from opinions, the relevant from the non-relevant, as well as fantasy from reality in problem-solving experiences. Creative thinking, too, is necessary for determining new methods of solving a problem. Novel, unique ways of viewing and attempting solutions at problem-solving are to be encouraged.

Mathematics can be made into a fascinating academic discipline with the solving of lifelike problems whereby students may see its use in society. Challenging and interesting methods may be used to stimulate student's interest in learning mathematics. Mathematics certainly need not be dull and uninteresting. It should be enjoyable and fascinating. New objectives need to be built upon what pupils have learned previously

There are diverse kinds of knowledge that teachers need to teach mathematics to students and include knowledge of:

- Subject-matter including important principles and meaning, formulas, facts, processes and procedures, rules, definitions, as well as structural ideas.
- Pedagogy such as lesson and unit planning and implementation, questions to ask and problem to solve, explanations to use, examples to provide as in diagrams and formulas, and demonstrations to make situations meaningful and concrete.
- Pedagogical subject-matter including determining prerequisites students possess before instruction, the sequence used to develop vital mathematical understandings among learners, diagnosis of student difficulties in learning, as well as strategies to use with quality materials of instructions.
- Information about students in the classroom. Here, the teacher needs to understand the developmental level of each student and where he/she is achieving presently. The teacher should not impart content too difficult, nor too easy for learner attainment. Thus, the subject matter to be imparted needs to be challenging, yet achievable.
- Curriculum development, curriculum practices and procedures. Mathematics teachers need to develop proficiency in writing cognitive, psychomotor, and affective objectives for student's attainment. These must be written in measurable terms to ascertain if they have/have not been achieved by learners. Diagnosis is then possible to determine which learning activities are necessary for the student to remedy deficits. A variety of valid, reliable measurement procedures needs to be used to determine the student's progress.

The mathematics teacher then must appraise the self to notice if there are personal strengths in mathematics to effectively emphasize the scope and sequences of needed math in the curriculum to teach effectively. Online education is also possible to work at home, at one's convenience, in taking additional course work in mathematics knowledge and pedagogy. School-sponsored workshops, and attending a professional meeting, are further avenues of in-service education. The principles of educational psychology must be stressed adequately in course work dealing with imparting mathematical knowledge to students.

A problem can be defined as

> *'.... any task or activity for which the students have no prescribed or memorized rules or methods, nor is there a perception by students that there is a specific "correct" solution method'* – **Hiebert et al.,1997**

Effective teachers will use such problems as starting points and an ongoing means for students to investigate and understand conceptual ideas so they can develop skills and procedures. Through such problems, all students are provided with appropriate entry points to progressively develop the understanding of concepts and increasingly more complex skills that facilitate efficient problem-solving.

Using mathematics to express ideas or to solve problems involves at least three phases: (1) representing some aspects of things abstractly, (2) manipulating the abstractions by rules of logic to find new relationships between them, and (3) seeing whether the new relationships say something useful about the original things.

As we have analysed the nature of mathematics and its implications for mathematics teaching we can list some of the important aspects to be considered while teaching mathematics

Mathematics should be taught or learned through:

- Constructing own understandings.
- Applying prior knowledge and skills.
- Encouraging risk-taking.
- By providing each student with challenges that meet their level through the careful use of investigative tasks.
- Creating purposeful learning experiences for students through the use of relevant and meaningful contexts.
- Acknowledging students' prior learning and help them make connections between what they already know and what they are currently learning.

Check Your Progress - 2

1. List the nature of mathematics which are useful in developing methods of teaching mathematics.
2. List the three phases of expressing ideas or solving problems in mathematics.

2.6.3.3Nature of science and its implications for science pedagogy

a. Scientific knowledge is tentative

Although it is reliable and durable, scientific knowledge is neither set in concrete nor perfect. Rather, it is subject to change in the light of new evidence or a new interpretation of existing evidence. Because of its tentative nature, we cannot claim 'absolute truth' in science. The tentative nature of scientific knowledge also means that laws and theories may change.

This tells that while teaching one needs to make clear that what they are learning under science need not be true forever. We need to give examples of how scientific truths have changed over time.

b. Science is a social activity

Scientific truths though sometimes have come to light in the name of one single scientist, it is not the product of the thinking of one individual. Each scientist has invented something unique as a result of interaction with many other scientists.

While teaching science, one can encourage one to realize scientific truths in association with others. Today many scientific projects are undertaken in groups and they come out with new truths. There are some fields of science where you can be successful with pure thought. Einstein was largely working on his own, based on conversations with other people.

c. The empirical nature of science

This means that science is based on and derived from observations of the world around us from which interpretations are made. Scientists depend on empirical evidence to produce scientific knowledge. Any scientific explanation must be consistent with empirical evidence, and new evidence brings the revision of scientific knowledge.

This has a very significant implication for teaching and learning of science. Methods of teaching science differ crucially from other school subjects because of this nature of science. One cannot make students learn a science unless you create a situation to observe directly or in any other mode, what they are learning. That is why science cannot be taught through storytelling or narrative method. For example, when a child tells his mother that " mamma, water consists of 2 molecules of hydrogen and one molecule of oxygen, the mother asks him, "how do you know my child"? The child should be in a position to say, " I observed with my eyes mamma, or I saw mamma" but not " my teacher said so".

We know that science is nothing but the discovery of nature. The key to this discovery is observation. Observation is one of the basic processes of science. Most of the scientific truths have been discovered through observation. Therefore one of the basic implications of this nature of science, for the teaching of science is, science should be taught through observation wherever it is needed. There is no other method than observation to teach topics like the structure of plants, grafting methods, etc.

d. The inferential, imaginative and creative nature of science

However, science isn't simply the accumulation of observable evidence and the orderly gathering of knowledge. All observations require interpretation and inference by scientists. To do this, scientists require imagination and creativity to make inferential statements about what they see. Imagination and creativity are needed in every aspect of a scientist's work – making sense of observations, making the creative leap from data to a possible explanation, coming up with new ideas, designing investigations, and looking at old data in a new light.

Therefore, students should be taught how to make inferences based on the available data. This means science teaching is characterised by the 'inductive approach' where the examples or data are presented to the students and the students are motivated to make generalisations or inference. Students need to be encouraged to be imaginative also. When the teacher is teaching about the solar system, a teacher cannot make students observe all the phenomena that the students are supposed to learn. Most of the deductions in astronomy are based on imagination. Students should be trained to be creative to understand science better.

Unless they are creative, they cannot understand the unique situations imagined, observed, or created by scientists. Students, to proceed in the lines of scientists need to be creative and only then they can contribute to the world of science.

e. The subjective and theory-laden nature of science

Different scientists can interpret the same datasets differently. How can this be so? Scientists do strive to be objective, but it is just not possible to make truly objective observations and interpretations without any bias. A scientist's mind is not a blank slate. Individual scientists have their prior knowledge, theoretical beliefs, experiences, cultural background, training, expectations, and biases, each of which will affect their observations and conclusions. All observation is preceded by theory and conceptual knowledge. Science tries to overcome this lack of pure objectivity through the scientific community, which scrutinizes scientific work and helps balance individual scientists' leanings.

Many of the teaching and learning activities could be used to demonstrate to students how much prior knowledge they bring to any scientific investigation.

f. The socially and culturally embedded nature of science

All scientific knowledge is produced within a larger society and culture. This means that the social and cultural elements such as politics, economics, power structures, religion, and philosophy will affect the scientific knowledge produced and how it is accepted. This also means that the direction and the products of science will be greatly influenced by the society and the culture in which the science is conducted.

As societies change, so do scientific priorities. For example, during the first half of the 20th century, two World Wars dominated society and so governments made funding available for research with wartime applications. Science moved in that direction and nuclear energy was unlocked. Science changes to reflect shifts in society and its priorities.

All scientific knowledge can also be seen to be embedded in a global scientific community. This community has a particular culture, expectations, and accumulated knowledge - all of which are essential to increasing scientific knowledge.

The teacher cannot make children learn science in isolation. The child needs to be open to the environment in which he/she lives and then only the scientific phenomena can be understood better. TO understand environmental science better, the child needs to be open to the present-day world. The present status, reasons for the present status, etc can be understood only from a social and cultural background.

As we have analysed the nature of science and its implications for science teaching we can list some of the important aspects to be considered while teaching science?
Science should be taught or learned through

- Inductive method
- Observation
- Making Inferences
- Experimentation
- Creative ways
- Collaborative efforts
- Social interaction
- Understanding culture

Check Your Progress - 3
Explain the nature of science that would decide the implications for science pedagogy

2.6.3.4. Pedagogical aspects of Science and Mathematics

Math is not science. Sciences seek to understand some aspect of phenomena, and is based on empirical observations, while math seeks to use logic to understand and often prove relationships between quantities and objects which may relate to no real phenomena. Scientific theories may be supported by evidence, but not proven, while we can prove things in math. Let us present the pedagogical aspects of science and mathematics in the table below:

Sl. No.	Teaching of Science	Teaching of Mathematics
1	Inductive: Starts from examples and generalizes the end.	Deductive: Starts from a generalization and deducts the answer at the end.
2	Making the scientific truths clear.	Making the abstract concrete. Flexible ways of thinking with repeated practice over time.
3	Observation and Experimentation are major methods.	Logical deduction or hypothetical deduction is the major method.
4	Learning through observation and experimentation.	Learning through practice, and "drill the basics.
5	Sense experience is important.	Intellectual and intuitive thinking is important.

Check Your Progress - 4
Explain how and why the pedagogical aspects of Science and Mathematics differ.

2.6.4. Let us Summarise

The lack of knowledge, concepts, formulas and basics and also lack of practice makes learning and teaching mathematics difficult. Mathematics is the science that deals with the logic of shape, quantity and arrangement. Mathematics is all around us, in everything we do. It is the building block for everything in our daily lives, including mobile devices, architecture (ancient and modern), art, money, engineering, and even sports.

To understand how mathematics is different from science, we not the following important points.

- Mathematics is poetry, while science is prose.
- Mathematics is an inner experience, while science deals with external experiences.
- Mathematics is an inner experience; it is as close to meditation as one gets. And that's the sheer delight of doing it.
- Mathematics is also of course deeply challenging intellectually. But science is messy. The challenges are different - not of the pure intellect sort.

Think of Mathematics as a Language and think of Science as one sentence within that Language. There are many sentences that you can create given a Language, but you cannot create a Language with many sentences. Thus, the following are some important points to be noted which make teaching mathematics different than science.

- Mathematics requires different study processes. In other courses, you learn and understand the material, but you hardly ever have to apply it. You have to do the problems to learn mathematics.

- Mathematics is a linear learning process. What is used one day is used the next, and so forth.
- Mathematics is much like a foreign language. It must be practiced every day, and often the vocabulary is unfamiliar.

One can't learn swimming without getting into the water. No one can climb a tree without practice. No one becomes a good artist without actually practicing it. The same way mathematics cannot be learned without doing it. This makes teaching mathematics different than teaching science.

2.6.5. Answer to 'Check Your Progress - 1, 2, 3 and 4'

Check Your Progress - 1
Refer Section 2.6.3.1.

Check Your Progress - 2
Refer Section 2.6.3.2.

Check Your Progress - 3
Refer Section 2.6.3.3.

Check Your Progress - 4
Refer Section 2.6.3.4

2.6.6. Unit end Exercises

1. Explain the relations between mathematics and science.
2. Describe the nature of mathematics its implications for mathematics pedagogy.
3. Describe the Nature of science and its implications for science pedagogy.
4. What are the major differences between teaching mathematics and teaching science? Explain.

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