Reg. No. | | | | | | | | | |

MTH 452

II Semester M.Sc. Degree Examination, Sept./Oct. 2022
MATHEMATICS
Algebra – II
(CBCS – New Syllabus)

Time : 3 Hours Max. Marks : 70

Notes : *1) Answer **any five full** questions.*
*2) **No** additional sheets will be provided for answering.*
*3) **Use** of scientific calculator is permitted.*

1. a) Let D be an integral domain. For a, b $\in$ D, prove or disprove that :
   i) a divides b in D if and only if $\langle a \rangle \subseteq \langle b \rangle$.
   ii) a is irreducible if and only if $\langle a \rangle$ is a maximal ideal in D.
   iii) $\langle a \rangle$ and $\langle b \rangle$ are equal if and only if a and b are associates in D.

   b) Define a Euclidean domain and a principal ideal domain. Prove that every Euclidean domain is a principal ideal domain.

   c) Let a, b be elements of a principal ideal domain R, not both zero. Prove that greatest common divisor of a and b exists in R and it is unique up to associates. **(6+4+4)**

2. a) Let D be an integral domain. Then prove that the factoring into irreducible terminates in D if and only if D does not contain infinite strictly increasing chain $\langle a_1 \rangle \not\subseteq \langle a_2 \rangle \not\subseteq \langle a_3 \rangle \not\subseteq \ldots$ of principal ideals.

   b) Prove that, the product of any two primitive polynomials is primitive.

   c) Find all irreducible polynomials of degree less than 4 in $\mathbb{Z}_2[x]$. **(5+5+4)**

3. a) Let f be an integer polynomial with positive leading coefficient. Then prove that f is irreducible in $\mathbb{Z}[x]$ if and only if it is either a prime integer or a primitive polynomial that is irreducible in $\mathbb{Q}[x]$.

   b) Let α be Gauss prime and $\overline{\alpha}$ be its complex conjugate. Then prove that $\alpha\overline{\alpha}$ is either a prime integer or square of a prime integer.

   c) Factor 6 + 9i into primes in $\mathbb{Z}[i]$. **(5+6+3)**

4. a) Prove or disprove the following :

   i) Every finite extension is an algebraic extension.

   ii) Every algebraic extension is a finite extension.

   b) If L is a finite extension of K and K is a finite extension of F, then prove that L is a finite extension of F.

   c) Let $K | F$ be a field extension and $\alpha \in K$. Then prove that α is algebraic over F, if and only if $F[\alpha] = F(\alpha)$. **(6+4+4)**

5. a) Show that trisecting an angle is impossible using Ruler and Compass alone.

   b) If p is a prime number and a regular p-gon can be constructed by using ruler and compass, then show that $p = 2^k + 1$, for some integer $k \geq 0$.

   c) Find the splitting field and the degree of extension of the splitting field of $f(x) = x^4 + x^2 + 1$ over $\mathbb{Q}$. **(6+4+4)**

6. a) Prove that characteristic of any finite field is a prime p and it has p^n elements, where n is a positive integer.

   b) Let F be a field of order p^n, where p is prime and n is a positive integer. Prove that F contains a subfield of order p^k if and only if $k | n$.

   c) If F is a field of characteristic zero and f(x) is an irreducible polynomial in F[x], then prove that f has no multiple root in any extension of F. **(5+5+4)**

7. a) Define an algebraically closed field. Prove that a field F is algebraically closed if and only if every polynomial in F[x] splits into linear factors in F[x].

   b) Let F be a field of characteristic zero. Then prove that any finite extension of F is a simple extension.

   c) Show that $\mathbb{Q}\left(i, \sqrt[3]{2}\right) = \mathbb{Q}\left(i \sqrt[3]{2}\right)$. Find $\left[\mathbb{Q}\left(i, \sqrt[3]{2}\right) : \mathbb{Q}\right]$. **(3+7+4)**

8. a) Define the fixed field of group of automorphisms of a field K. If K is a Galois extension of field F, then prove that the fixed field of Galois group G (K/F) is F.

   b) State and prove the main theorem of Galois theory. **(3+11)**
