Reg. No. | | | | | | | | | |

MTH 453

II Semester M.Sc. Degree Examination, September/October 2022
MATHEMATICS
Real Analysis – II
(CBCS – New Syllabus)

Time : 3 Hours Max. Marks : 70

Note : *1) Answer **any five full** questions.*
*2) **No** additional sheets will be provided for answering.*
*3) Use of scientific calculator is **permitted**.*

1. a) Let f be a bounded real function on [a, b] and α be a monotonically increasing real function on [a, b]. If f monotonic on [a, b], then prove that $f \in \mathcal{R}(\alpha)$ on [a, b].

   b) Let f be a bounded real function on [a, b] and α be a monotonically increasing real function on [a, b]. If f has only finitely many discontinuities on [a, b] and α is continuous at every point of discontinuity of f on [a, b], then prove that $f \in \mathcal{R}(\alpha)$ on [a, b]. **(5+9)**

2. a) Suppose $\sum c_n$ converges, where $c_n \geq 0$ for $n \in \mathbb{N}$, $\{s_n\}$ is a sequence of distinct points in (a, b) and $\alpha(x) = \sum c_n I(x - s_n)$, where I denotes the unit step function. Let f be continuous on [a, b]. Then prove that $\int_a^b f\, d\alpha = \sum c_n f(s_n)$.

   b) If γ' is continuous on [a, b], then show that γ is rectifiable and $A(\gamma) = \int_a^b |\gamma'(t)|\, dt$.

   c) Let f be a bounded real function on [a, b] and suppose $f^2 \in \mathcal{R}$ on [a, b]. Does it follow that $f \in \mathcal{R}$ on [a, b] ? Justify. **(6+6+2)**

3. a) Let f be a bounded real function on [a, b] and α be a monotonically increasing real function on [a, b]. Suppose $\alpha' \in \mathcal{R}$ on [a, b]. Prove that $f \in \mathcal{R}(\alpha)$ on [a, b] if and only if $f\alpha' \in \mathcal{R}$ on [a, b] and in that case show that $\int_a^b f\, d\alpha = \int_a^b f(x)\alpha'(x)dx$.

   b) Suppose $f \geq 0$ is a continuous function on [a, b] such that $\int_a^b f(x)\, dx = 0$. Prove that f = 0 on [a, b]. **(9+5)**

P.T.O.

4. a) Suppose f ≥ 0 and monotonically decreasing on $[1, \infty)$. Then prove that $\int_1^{\infty} f\,dt$ converges if and only if $\sum f(n)$ converges.

   b) Show that $f(x) = \frac{\sin x}{x}$, $x \in [1, \infty)$ is integrable on $[1, \infty)$, but not absolutely. **(8+6)**

5. a) Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x_0 be a limit point of E and suppose that $\lim_{t \to x_0} f_n(t) = A_n$ for all n. Then prove that $\{A_n\}$ converges and $\lim_{t \to x_0} f(t) = \lim_{n \to \infty} A_n$.

   b) Let α be monotonically increasing on [a, b]. Suppose $f_n \in \mathscr{R}(\alpha)$ on [a, b], for n = 1, 2, 3, ..., and suppose $f_n \to f$ uniformly on [a, b]. Then show that $f \in \mathscr{R}(\alpha)$ on [a, b], and $\int_a^b f\,d\alpha = \lim_{n \to \infty} \int_a^b f_n d\alpha$. **(8+6)**

6. a) Prove that there exists a real continuous function on the real line which is nowhere differentiable.

   b) If K is a compact metric space, if $f_n \in \mathscr{C}(K)$ for all $n \geq 1$, and if $\{f_n\}$ converges uniformly on K, then show that the family $\tau = \{f_n\}$ is equicontinuous on *K*. **(11+3)**

7. a) If f is a complex valued continuous function on [a, b], then prove that there exists a sequence $\{P_n\}$ of polynomials with complex coefficients such that $\lim_{n \to \infty} P_n(x) = f(x)$ uniformly on [a, b].

   b) If f is continuous on [0, 1] and if $\int_0^1 f(x)\, x^n\, dx = 0$ for all n = 1, 2, 3, . . ., then prove that $f(x) = 0\ \forall x \in [0, 1]$. **(12+2)**

8. a) Suppose f maps an open subset $E \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$. Then $f \in \mathscr{C}'(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \leq i \leq m$, $1 \leq j \leq n$.

   b) If X is a complete metric space, and if ϕ is a contraction of X into X, then prove that there exists one and only one $x \in X$ such that $\phi(x) = x$. **(10+4)**