Reg. No. | | | | | | | | | |

MTH 454

II Semester M.Sc. Degree Examination, September/October 2022
(CBCS – New Syllabus)
MATHEMATICS
Topology

Time : 3 Hours Max. Marks : 70

Note : *1) Answer **any five full** questions.*
*2) **No** additional sheets will be **provided** for answering.*
*3) **Use** of scientific calculator is **permitted**.*

1. a) Show that $\tau = \{\mathbb{R}\} \cup \{G \subseteq \mathbb{R} : 0 \notin G\}$ is a topology on the set $\mathbb{R}$ of all real numbers.

   b) If there exists a homeomorphism of a topological space X onto a metric space Y, then show that X is metrizable.

   c) Define closure $\overline{A}$ of a set A in a topological space X. Show that $\overline{A}$ is the smallest closed set containing the set A. If $\mathbb{N}$ denotes the set of all positive integers, then find $\overline{\mathbb{N}}$ in the space $(\mathbb{R}, \tau)$, where τ is given in the problem 1 (a). **(3+6+5)**

2. a) Let A be a subset of a topological space X . Prove that a point $x \in \overline{A}$ if and only if each neighborhood of x intersects A.

   b) Let $f : X \to Y$ be a mapping of one topological space into another. Show that the following statements are equivalent :
      i) $f^{-1}(F)$ is closed in X whenever F is closed in Y .
      ii) $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X.

   c) Prove or disprove: If X is a non-empty set, Y is a topological space, and $f : X \to Y$ is any map, then $\{f^{-1}(V) : V \text{ is open in } Y\}$ is a topology on X. **(5+6+3)**

3. a) If S is a class of subsets of a non-empty set X, show that S can serve as an open subbase for a topology on X .

   b) Show that $\overline{X - A} = X - \text{Int}(A)$ for every subset A of a topological space.

   c) Prove or disprove : Every second countable space is first countable. **(7+4+3)**

4. a) Define the weak topology generated by a class of functions. Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 0 & \text{if} \quad x \le 1 \\ 2 & \text{if} \quad x > 1 \end{cases}$ $\forall x \in \mathbb{R}$. Give a complete description of the weak topology on $\mathbb{R}$ generated by the function f.

   b) Let f be a mapping of a topological space X into a product space $\prod_{i \in I} X_i$. Show that f is continuous if and only if $\pi_i \circ f : X \to X_i$ is continuous for each projection π_i.

   c) Prove that a topological space is compact if and only if every class of closed sets with the finite intersection property has non-empty intersection. **(4+5+5)**

5. a) Show that every closed and bounded interval [a, b], a, b $\in \mathbb{R}$, of the real line $\mathbb{R}$ is compact.

   b) State and prove Tychonoff's theorem.

   c) Prove or disprove : Every compact subspace of a topological space is closed. **(6+6+2)**

6. a) Prove that a topological space X is a T_1-space if and only if every finite set is closed in X.

   b) Let $\{X_i : i \in I\}$ be a non-empty class of topological spaces. Prove that the product space $\prod_{i \in I} X_i$ is Hausdorff if and only if X_i is Hausdorff for each $i \in I$.

   c) Prove or disprove : A finite T_1-space is discrete. **(6+6+2)**

7. a) Prove that every compact Hausdorff space is normal.

   b) Let X be a first countable space, Y be a topological space, $f : X \to Y$ be a map. Prove that the following statements are equivalent :

      i) $f : X \to Y$ is continuous.

      ii) For every sequence $x_n \to x$ in X, $f(x_n) \to f(x)$ in Y.

   c) State Urysohn's Lemma. **(6+6+2)**

8. a) Show that a topological space X is disconnected if and only if there exists a continuous mapping of X onto the discrete two-point space {0, 1}.

   b) Prove that the product of any non-empty class of connected spaces is connected. **(5+9)**
