Reg. No. | | | | | | | | | |

MTH 552

IV Semester M.Sc. Degree Examination, Sept./Oct. 2022
(CBCS – New Syllabus)
MATHEMATICS
Complex Analysis – II

Time : 3 Hours   Max. Marks : 70

Notes : *1) Answer **any five full** questions.*

*2) **No** additional sheets will be provided for answering.*

*3) **Use** of scientific calculator is **permitted**.*

*4) **All** questions carry **equal** marks.*

1. a) If f(z) is meromorphic in Ω with the zeros a_j and the poles b_k, then show that $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k)$ for every cycle γ which is homologous to zero in Ω and does not pass through any of the zeros or poles.

   b) How many roots of the equation $z^4 - 6z + 3 = 0$ have their modulus between 1 and 2 ?

   c) Find the poles and residues of the following function

      i) $\frac{1}{\sin^2 z}$

      ii) $\frac{1}{z^m(1-z^n)}$

      where m, n $\in \mathbb{Z}_+$. **(7+3+4)**

2. a) Suppose that f(z) is analytic in the annulus $r_1 < |z| < r_2$ and continuous on the closed annulus. If M(r) denote the maximum of |f(z)| for |z| = r, show that $M(r) \le M(r_1)^\alpha M(r_2)^{1-\alpha}$ where $\alpha = \log(r_2/r) : \log(r_2/r_1)$.

   b) Derive Poisson's formula for harmonic function. **(7+7)**

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3. a) If the function $f_n(z)$ are analytic and $\neq 0$ in a region Ω and if $f_n(z)$ converges to f(z), uniformly on every compact subset of Ω, then show that f(z) is either identically zero or never equal to zero.

b) Show that the series

$\zeta(z) = \sum_{n=1}^{\infty} n^{-s}$ converges for Re z > 1, and represents its derivatives in series form.

c) What is the co-efficient of z^7 in the Taylor development of tan z ? Also develop log (sin z/z) in powers of z upto the term z^5. **(6+3+(3+2))**

4. a) Show that $\dfrac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{+\infty} \dfrac{1}{(z-n)^2} + g(z)$ where g(z) is analytic in the whole complex plane.

b) Prove that

$$\sin \pi(z+\alpha) = \sin(\pi\alpha)\, e^{\pi z \cot \pi\alpha} \prod_{-\infty}^{+\infty} \left(1 + \frac{z}{n+\alpha}\right) e^{-z/(n+\alpha)}.$$ **(7+7)**

5. a) Obtain Jensen's formula.

b) Using Poisson's formula derive Poisson-Jensen's formula. **(10+4)**

6. a) Show that there exists a basis (w_1, w_2) such that the ratio $r = w_2/w_1$ satisfies the following conditions :

i) $\text{Im}\, \gamma > 0$

ii) $-\frac{1}{2} < \text{Re}\, \gamma \leq \frac{1}{2}$

iii) $|\gamma| \geq 1$

iv) $\text{Re}\, \gamma \geq 0$ if $|\gamma| = 1$.

Further show that the ratio γ is uniquely determined by these conditions and there is a choice of two, four, or six corresponding bases.

b) Show that an elliptic function without poles is a constant function. **(10+4)**

7. a) Prove that $\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$ converges absolutely and uniformly on every compact subset of $\mathbb{C}$.

   b) Show that the function $\theta(z) = \prod_{n=1}^{\infty}\left(1+h^{2n-1}e^{z}\right)\left(1+h^{2n-1}e^{-z}\right)$ where $|h| < 1$ is analytic in the whole plane and satisfies the functional equation $\theta(z + z \log h) = h^{-1} e^{-z} \theta(z)$. **(6+8)**

8. a) Show that any even elliptic function for which '0' is neither a zero nor a pole can be written as :

$$C\prod_{n=1}^{n}\frac{P(z)-P(a_k)}{P(z)-P(b_k)}.$$

   b) Show that $P(z)-P(u)=\frac{-\sigma(z-u)\sigma(z+u)}{(\sigma+z)^2(\sigma(u))^2}$. **(10+4)**
