Reg. No. | | | | | | | | | |

MTH 553

IV Semester M.Sc. Degree Examination, September/October 2022
(CBCS – New Syllabus)
MATHEMATICS
Functional Analysis

Time : 3 Hours   Max. Marks : 70

Note : *1) Answer **any five full** questions.*
*2) Answer to **each full** question shall **not** exceed **eight** pages of the answer book. **No** additional sheets will be provided for answering.*
*3) Use of scientific calculator is **permitted**.*

1. a) Let N be a normed linear space and let M be a closed linear subspace of N. For $x + M \in N/M$, define $||x + M|| = \inf \{||x + m|| : m \in M\}$. Show that N/M is a normed linear space. Further show that if N is a Banach space, then so is N/M.

   b) Prove that the interior of a proper linear subspace of a normed linear space is empty. **(10+4)**

2. a) Let $T : N \to N'$ be a linear transformation of normed linear spaces. Show that the following are equivalent :

   i) T is continuous

   ii) T is continuous at the origin

   iii) there exists a $K \geq 0$ such that $||T(x)|| \leq K||x||$ for all $x \in N$.

   iv) if $S = \{x \in N : ||x|| \leq 1\}$, then T(S) is a bounded subset of N′.

   b) Give an example of a linear transformation of normed linear spaces which is not continuous.

   c) Let M be a closed linear subspace of a normed linear space N and let $x_0 \in N - M$. Prove that there exists a functional $f_0 \in N^*$ such that $f_0(M) = 0$ and $f_0(x_0) \neq 0$. **(6+4+4)**

3. a) Let M be a linear subspace of a normed linear space N and let f be a functional defined on M. If x_0 is a vector not in M and if $M_0 = M + [x_0]$ is the linear subspace spanned by M and x_0, then show that f can be extended to a functional f_0 on M_0 such that $||f_0|| = ||f||$.

   b) Let a Banach space B be made into a Banach space B′ by means of a new norm. Show that the topologies generated by these norms are same if either is stronger than the other. **(12+2)**

4. a) Let B and B′ be Banach spaces and let $T : B \to B'$ be a linear transformation. Then show that T is continuous if its graph is a closed set in $B \times B'$.

   b) State and prove the Uniform Boundedness theorem. Hence prove that, a non-empty subset X of a normed linear space N is bounded if and only if f(X) is a bounded set of scalars for every $f \in N^*$. **(5+9)**

5. a) State and prove the Schwarz inequality for a Hilbert space H and hence derive that the inner product on H is jointly continuous.

   b) Is the Banach space $C([0, 1], \mathbb{R})$ of all continuous real functions on [0, 1] with sup norm, a Hilbert space ? Justify.

   c) If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, then prove that the linear subspace M + N is also closed.

   d) If S is a non-empty subset of a Hilbert space H, then show that $S^{\perp} = S^{\perp\perp\perp}$. **(5+3+4+2)**

6. a) If $\{e_i\}$ is an orthonormal set in a Hilbert space H and if $x \in H$, then prove that $x - \sum \langle x, e_i \rangle e_i$ is orthogonal to e_j for all j.

   b) Show that every orthonormal set in a Hilbert space H is contained in some complete orthonormal set. Use it to prove the following :
   If M is a proper closed linear subspace of H, then there exists a non-zero vector z_0 in H such that $z_0 \perp M$. **(10+4)**

7. a) Prove that the following statements are equivalent for an orthonormal set $\{e_i\}$ of a Hilbert space H :
      i) $\{e_i\}$ is complete.
      ii) $x \perp e_i$ for all i implies x = 0.
      iii) If x is an arbitrary vector in H, then $x = \sum \langle x, e_i \rangle e_i$.
      iv) If x is an arbitrary vector in H, then $||x||^2 = \sum |\langle x, e_i \rangle|^2$.

   b) Prove that every Hilbert space is reflexive.

   c) Show that an operator T on a Hilbert space H is normal if and only if $||T^*x|| = ||Tx||$ for all $x \in H$. Hence prove that if N is a normal operator on H, then $||N^2|| = ||N||^2$. **(6+4+4)**

8. a) Let M be a closed linear subspace of a Hilbert space H, P be a projection on M and T be an operator on H. Then prove the following :

   i) M is invariant under T if and only if TP = PTP.

   ii) M reduces T if and only if TP = PT.

   b) If P and Q are projections on closed linear subspaces M and N of a Hilbert space H, then prove that the following statements are equivalent :

   i) $P \leq Q$

   ii) $||Px|| \leq ||Qx||$ for every $x \in H$

   iii) $M \subseteq N$

   iv) PQ = P

   v) QP = P.

   c) Let $P_1, \ldots, P_n$ be projections on the closed linear subspaces $M_1, \ldots, M_n$ of a Hilbert space H and $M = M_1 + \ldots + M_n$. Then show that $P = P_1 + \ldots + P_n$ is a projection if and only if P_i's are pairwise orthogonal and in this case P is the projection on M. **(4+4+6)**
