Reg. No. | | | | | | | | | |

MTS 456

II Semester M.Sc. Degree Examination, September/October 2022
MATHEMATICS
Ordinary Differential Equations
(CBCS – New Syllabus)

Time : 3 Hours Max. Marks : 70

Note : *1) Answer **any five full** questions.*
*2) **No** additional sheets will be provided for answering.*
*3) **Use** of scientific calculator is **permitted**.*

1. a) If ϕ is any solution of $L_n(y) = y^{(n)} + a_1 y^{(n-1)} + ... + a_n y = 0$ on an interval I containing a point x_0, then prove that $\forall x$ in I, $e^{-k|x-x_0|}||\phi(x_0)|| \leq ||\phi(x)|| \leq ||\phi(x_0)|| e^{k|x-x_0|}$ where, $||\phi(x)|| = [|\phi(x)|^2 + |\phi^1(x)|^2 + ... + |\phi^{n-1}(x)|^2]^{1/2}$ and $K = 1 + |a_1| + |a_2| + ... + |a_n|$.

   b) Let $\phi_1(x)$ and $\phi_2(x)$ be linearly independent solutions of $L_2(y) = 0$ on an interval I. Then prove that every solution of $L_2(y) = 0$ can be expressed uniquely as $\phi(x) = c_1\phi_1(x) + c_2\phi_2(x)$, where c_1 and c_2 are constants.

   c) Let $\alpha_1, \alpha_2, ... \alpha_n$ be any n constants and let x_0 be any real number. Prove that on any interval I containing x_0 there exists at most one solution ϕ of $L_n(y) = 0$ satisfying $\phi(x_0) = \alpha_1, \phi^1(x_0) = \alpha_2, ... \phi^{(n-1)}(x_0) = \alpha_n$. **(7+4+3)**

2. a) Compute the Wronskian of $y''' - 4y' = 0$. Verify that they are linearly independent.

   b) Use method of variation of parameters to find a particular solution of the equation

      i) $x^2y'' - 2xy' + 2y = 6x^4$

      ii) $y'' + y = \sec x$.

   c) Find the solution ϕ of the initial-value problem
   $y''' + y = 0, y(0) = 0, y'(0) = 1, y''(0) = 0$. **(4+4+6)**

3. a) Solve the following equation by the method of undetermined coefficients :

      i) $y'' - 2y' + y = 2e^x + 2x$

      ii) $y'' + y = xe^x \cos 2x$.

   b) If ϕ_1 is a solution of $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$, on an interval I and $\phi_1(x) \neq 0 \forall x \in I$, then show that $\phi_2(x) = \phi_1(x)\int_{x_0}^{x} \frac{1}{[\phi_1(s)]^2} \text{Exp}\left[\int_{x_0}^{s} \frac{-a_1(\xi)}{a_0(\xi)} d\xi\right] ds$ is another solution. **(8+6)**

P.T.O.

4. a) Find the power series solution of $y'' + 3xy' + 3y = 0$, $y(0) = 2$, $y'(0) = 3$.

   b) Prove the following recurrence relations :

      i) $(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$

      ii) $nP_n = xP'_n - P'_{n-1}$

   where $P_n(x)$ is the Legendre polynomial. **(8+6)**

5. a) Show that $\int_{-1}^{1} P_n(x)P_m(x)dx = \begin{cases} 0, & m \neq n \\ \dfrac{2}{2m+1}, & m = n \end{cases}$ where $P_n(x)$ represents the Legendre polynomials.

   b) Obtain the general solution of $(x^2 + 1)y'' + xy' - xy = 0$ about regular singular point $x = 0$. **(7+7)**

6. a) Using Frobenius method, find the series solution of $2x^2y'' + x(2x + 1)y' - y = 0$ near the singular point $x = 0$.

   b) Prove that $\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$. **(10+4)**

7. a) State and prove existence and uniqueness theorem for the solution of the initial value problem $y' = A(x)y$, $y(x_0) = y_0$.

   b) Solve the system of equations $\begin{pmatrix} y_1^1 \\ y_2^1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. **(7+7)**

8. a) Let f be a continuous real valued function on the rectangle $R : |x - x_0| \leq a, |y - y_0| \leq b, (a, b > 0)$ and let $|f(x, y)| \leq M$ for all (x, y) in R. Further suppose that f satisfies a Lipschitz condition with constant K in R. Then prove that the successive approximations
   $\phi_0(x) = y_0, \ \phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\, dt, k = 0, 1, 2, ...$
   converges on the interval $I : |x - x_0| \leq \alpha = \min\left\{a, \frac{b}{M}\right\}$ to a solution ϕ of the IVP $y' = f(x, y)$, $y(x_0) = y_0$ on I.

   b) Apply Picards method to solve the IVP upto 3[rd] approximation $\frac{dy}{dx} = x + y^2$ given that $y = 0$ at $x = 0$. **(9+5)**
