Reg. No. | | | | | | | | | |

MTS 557

IV Semester M.Sc. Degree Examination, September/October 2022
MATHEMATICS
Algebraic Number Theory
(CBCS – New Syllabus)

Time : 3 Hours Max. Marks : 70

Note : *1) Answer* ***any five full*** *questions.*
2) Answer to ***each full*** *question shall* ***not*** *exceed* ***eight*** *pages of the answer book.* ***No*** *additional sheets will be* ***provided*** *for answering.*
3) ***Use*** *of scientific calculator is* ***permitted****.*

1. a) State and prove Liouville's theorem for a real algebraic number of degree more than 1.

   b) Define the notions of trace and norm of an element in algebraic number field K. If K is an algebraic number field of degree n over $\mathbb{Q}$, then show that for any, $\alpha \in K$, $Tr_K(\alpha) = \sigma_1(\alpha) + \sigma_2(\alpha) + \ldots + \sigma_n(\alpha)$, and $N_K(\alpha) = \sigma_1(\alpha)\,\sigma_2(\alpha)\ldots\sigma_n(\alpha)$, where $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the n distinct $\mathbb{Q}$-isomorphisms of K into $\mathbb{C}$. Deduce that for any algebraic integer α, $Tr_K(\alpha)$ and $N_K(\alpha)$ are integers.

   c) If $\xi = e^{\frac{2\pi i}{7}}$ and $K = \mathbb{Q}(\xi)$, then find $T_{K/\mathbb{Q}}\left(\xi + \xi^2 + \xi^3\right)$. **(5+7+2)**

2. a) Find the units in the ring of algebraic integers $\mathcal{O}_K$, where $K = \mathbb{Q}\left(\sqrt{d}\right)$ and d is a negative square free integer.

   b) Prove that every algebraic number field has an integral basis.

   c) Determine the discriminant of $K = \mathbb{Q}\left(\sqrt{d}\right)$, where d is a square-free integer. **(6+5+3)**

3. a) Prove that factorization into irreducibles is not unique in the ring of integers of $\mathbb{Q}\left(\sqrt{-5}\right)$.

   b) Show that $K = \mathbb{Q}\left(\sqrt{d}\right)$ is norm-Euclidean for d = – 1, – 2, – 3, – 7 and – 11.

   c) Show that $x^2 + 5 = y^3$ has no integer solution. **(5+5+4)**

P.T.O.

4. Let K be an algebraic number field and let $\mathcal{O}_K$ be its ring of integers.

   a) If norm of an element x in $\mathcal{O}_K$ is a rational prime, prove that x is irreducible in $\mathcal{O}_K$.

   b) Prove that $\mathcal{O}_K$ is a Dedekind domain. **(2+12)**

5. a) Let $\mathcal{O}_K$ be the ring of integers of an algebraic number field K. Let P be a non-zero prime ideal of $\mathcal{O}_K$ and let P^{-1} denotes the set $\{\alpha \in K : \alpha P \subseteq \mathcal{O}_K\}$. Show that P^{-1} is a fractional ideal of $\mathcal{O}_K$, $\mathcal{O}_K \subseteq P^{-1}$ and $PP^{-1} = \mathcal{O}_K$.

   b) Find the inverse of ideal $\langle 2, \sqrt{6} \rangle$ in $D = \mathbb{Z} + \mathbb{Z}\sqrt{6}$. **(10+4)**

6. a) If I and J are ideals of a Dedekind domain R, prove that I divides J if and only if $J \subseteq I$. Hence deduce that if P is prime ideal and P|IJ, then P|I or P|J.

   b) Prove or disprove the following :

      i) Every unique factorization domain is integrally closed.

      ii) Every principal ideal domain is a Dedekind domain.

      iii) Every Dedekind domain is a principal ideal domain.

   c) Find the prime factorization of the ideal $68\mathbb{Z}$ in the ring $\mathbb{Z}$. **(4+8+2)**

7. a) Let K be an algebraic number field and let $\mathcal{O}_K$ be its ring of integers. Prove that any non-zero proper ideal of $\mathcal{O}_K$ has finitely many divisors.

   b) Let K be an algebraic number field of degree n. If a rational prime p ramifies in K, then prove that p divides d_K, the discriminant of K.

   c) Let $K = \mathbb{Q}(\theta)$, where θ is a root of the polynomial $x^3 - 18x - 6$. Find out how the rational primes 2, 3, 5 and 11 split in K. **(5+5+4)**

8. a) Define the class-group and the class-number of an algebraic number field K. Prove that $\mathcal{O}_K$, the ring of integers of K is unique factorization domain if and only if the class-number is 1.

   b) Determine the class number of $K = \mathbb{Q}\left(\sqrt{-19}\right)$. **(8+6)**
